MATH2004 Tutorial 4

1. Find the domain of $f(x,y) = \sqrt{y - 2x^2}$.

   Solution: $y - 2x^2 \geq 0$, $y \geq 2x^2$.

2. Find the limit
$$\lim_{(x,y)\to(1,0)} \frac{x+y}{x^2+y^2+1}.$$

   Solution:
$$\lim_{(x,y)\to(1,2)} \frac{x+y}{x^2+y^2+1} = \frac{3}{6} = \frac{1}{2}.$$

3. Find the limit
$$\lim_{(x,y)\to(0,0)} \frac{x}{\sin y}.$$

   Solution: If we take the route $y = x$, then
$$\lim_{(x,y)\to(0,0)} \frac{x}{\sin y} = \lim_{x\to 0} \frac{x}{\sin x} = 1.$$
   If we take the route $y = 2x$, then
$$\lim_{(x,y)\to(0,0)} \frac{x}{\sin y} = \lim_{x\to 0} \frac{x}{\sin 2x} = \frac{1}{2} \lim_{x\to 0} \frac{2x}{\sin 2x} = \frac{1}{2}.$$
   The two limits are different, the original limit does not exist.

4. Let $f(x,y) = e^{xy} + x\ln(1+y^2)$. Calculate $f_x(0,1)$, $f_y(0,1)$, $f_{xy}(0,1)$.

   Solution:
$$f_x = ye^{xy} + \ln(1+y^2), \quad f_x(0,1) = 1 + \ln 2.$$
$$f_y = xe^{xy} + \frac{2xy}{1+y^2}, \quad f_y(0,1) = 0.$$
$$f_{xy} = (1+xy)e^{xy} + \frac{2y}{1+y^2}, \quad f_{xy}(0,1) = 2.$$

5. Find $\frac{\partial z}{\partial x}$, if z is implicitly defined by

$$x^2 + y^3 + z^4 - 8xyz = z^2.$$

Solution:

$$2x + 0 + 4z^3 z_x - 8yz - 8xyz_x = 2zz_x, \Rightarrow z_x = \frac{2x - 8yz}{2z - 4z^3 + 8xy} = \frac{x - 4yz}{z - 2z^3 + 4xy}.$$

6. $f(x, y, z) = \arctan(xy^2z^3)$. Find $f_z(x, y, z)$.

Solution:

$$f_z(x, y, z) = \frac{(xy^2z^3)_z}{1 + (xy^2z^3)^2} = \frac{3xy^2z^2}{1 + (xy^2z^3)^2}.$$

7. Let $f(x, y) = x^2 - xy$.

(1) Find the gradient of the function.

(2) Find the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction of the vector $\vec{v} = (3, -4)$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Solution: (1) $\nabla f = (2x - y, -x)$.

(2) $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = (\frac{3}{5}, -\frac{4}{5})$.

$D_{\vec{u}} f(2, 1) = (3, -2) \cdot (\frac{3}{5}, -\frac{4}{5}) = \frac{17}{5}$.

(3) The maximum rate of change of f at (2, 1) $= |\nabla f(2, 1)| = |(3, -2)| = \sqrt{13}$,

which occurs in the direction $\langle 3, -2 \rangle$.

8. Find the directional derivative of the function $f(x, y, z) = xz^2 - 2x^2y + 3y^2z + 9$ in the direction $\vec{u} = (\frac{3}{7}, -\frac{2}{7}, \frac{6}{7})$ at the point $(2, 1, -1)$.

Solution: $f_x = z^2 - 4xy$, $f_y = 2x^2 + 6yz$, $f_z = 2xz + 3y^2$. We have

$$f_x(2, 1, -1) = -7, f_y(2, 1, -1) = -14, f_z(2, 1, -1) = -1.$$

Note that $\vec{u}$ is a unit vector, thus

$$D_{\vec{u}} f(2, 1, -1) = (-7, -14, -1) \cdot (\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}) = \frac{1}{7}.$$