Name: **ID:**

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Probability Theory (MATH323A)

Mid-Term Exam, Monday October 24, 2016

NOTE: *This test consists of four questions. There is no optional question.*

Question 1. A local fraternity is conducting a raffle where 50 tickets are to be sold-one per customer. There are three prizes to be awarded. If the four organizers of the raffle each buy one ticket, what is the probability that the four organizers

(a) win all of the prizes? (*2 marks*)

Solution: Let X= the number of prizes which go to the organizers. Then

$$X \sim HyperGeometric(r = 4, n = 3, N = 50)$$

and hence

$$P(X = 3) = \frac{\binom{4}{3}\binom{50-4}{3-3}}{\binom{50}{3}} = \frac{\binom{4}{3}}{\binom{50}{3}} = 0.000204$$

(b) win exactly two of the prizes? (*3 marks*)

Solution:

$$P(X = 2) = \frac{\binom{4}{2}\binom{50-4}{3-2}}{\binom{50}{3}} = 0.014$$

(c) win none of the prizes? (*2 marks*)

Solution:

$$P(X = 0) = \frac{\binom{4}{0}\binom{50-4}{3-0}}{\binom{50}{3}} = 0.7745$$

Question 2. Suppose 5% of all people filing the long income tax form seek deductions that they know are illegal, while 2% incorrectly list deductions because they are unfamiliar with income tax regulations. Of the 5% who are guilty of cheating, 80% will deny knowledge of the error if confronted by an investigator. If the filer of the long form is confronted with an unwarranted deduction and he or she denies the knowledge of the error, what is the probability that he or she is guilty? (*6 marks*)

Solution: Define A={Seeking illegal deduction, i.e. guilty}, B={Incorrectly seek deduction}, and C= {No incorrect deduction }. Then $P(A) = 0.05$, $P(B) = 0.02$, and $P(C) = 0.93$. Let E={Confronted with an unwarranted deduction **and** deny knowledge of the error}. Thus $P(E \mid A) = 0.80$, $P(E \mid B) = 1$, and $P(E \mid C) = 0$. Note that $P(E \mid B) = 1$, since the filer is not seeking illegal deduction, and hence he/she is not aware of his/her mistake. On the other hand, $P(E \mid C) = 0$, since the filer will not be confronted with an unwarranted deduction. Note that E is the intersection of two events, {Confronted with an unwarranted deduction} and {deny knowledge of the error }. Now, using Bayes' theorem we have,

$$\begin{aligned} P(A \mid E) &= \frac{P(E \mid A)P(A)}{P(E \mid A)P(A) + P(E \mid B)P(B) + P(E \mid C)P(C)} \\ &= \frac{(0.8)(0.05)}{(0.8)(0.05) + (1)(0.02) + (0)(0.93)} \\ &= \frac{0.04}{0.06} \\ &= \frac{2}{3} \end{aligned}$$

Question 3. The mean number of automobiles entering a mountain tunnel per 2-minute period is one. An excessive number of cars entering the tunnel during a brief period of time produces a hazardous situation. Suppose that the Poisson model is reasonable. Assume that the tunnel is observed during ten 2-minute intervals, thus giving ten independent observations $Y_1, Y_2, \cdots, Y_{10}$. Find the probability that $Y > 3$ during at least one of the ten 2-minute intervals. (*6 marks*)

Solution: Let Y_i = the number of cars entering the tunnel over the i-th 2-minute interval for $i = 1, 2, \cdots, n$. Then Y_i for $i = 1, 2, \cdots, n$ are all independent and identically distributed according to *Poisson* distribution with $\lambda = 1$ as the mean number of autombile entering the tunnel over each 2-minute interval is 1. We first note that for $i = 1, 2, \cdots, 10$,

$$P(Y_i = k) = \frac{e^{-1}}{k!} \quad k = 0, 1, 2, \cdots$$

Thus

$$\begin{aligned}
P(Y_i > 3) &= 1 - P(Y_i \leq 3) \\
&= 1 - [P(Y_i = 0) + P(Y_i = 1) + P(Y_i = 2) + P(Y_i = 3)] \\
&= 1 - e^{-1}[1 + 1 + \frac{1}{2!} + \frac{1}{3!}] \\
&= 1 - \frac{8}{3}e^{-1} = 0.019 \quad .
\end{aligned}$$

Define, for $i = 1, 2, \cdots, 10$

$$X_i = \begin{cases} 1 & \text{if} \quad Y_i > 3 \\ 0 & \text{if} \quad Y_i \leq 3 \end{cases} \quad .$$

Then $P(X_i = 1) = p = 0.019$. Let $Z = \sum_{i=1}^{10} X_i$. We then have $Z \sim Bin(n = 10, p = 0.019)$. Therefore

$$\begin{aligned}
&P(Y > 3 \text{ during at least one of the ten 2-minute intervals}) \\
&\quad = P(Z \geq 1) = 1 - P(Z = 0) = 1 - (1-p)^{10} \\
&\quad = 1 - (0.981)^{10} = 0.1745 \quad .
\end{aligned}$$

Question 4. Two teams A and B play a series of games until one team wins four games. Assume that the games are played independently and that the probability that A wins any game is p. What is the probability that the series lasts exactly five games? (*6 marks*)

Solution: Define $X(Y)$ = number of games until team $A(B)$ wins team $B(A)$ four times. Then $X \sim NB(r = 4, p)$ and $Y \sim NB(r = 4, 1 - p)$. Now the series lasts exactly 5 games if either A wins the series at the 5th game or B does.

$$\begin{aligned} \alpha &= P(A \text{ wins the series at the 5th game}) \\ &= P(X = 5) = \binom{5-1}{4-1} p^4 (1-p)^{5-4} = 4p^4(1-p). \end{aligned}$$

Likewise,

$$\begin{aligned} \beta &= P(B \text{ wins the series at the 5th game}) \\ &= P(Y = 5) = \binom{5-1}{4-1} (1-p)^4 p^{5-4} = 4(1-p)^4 p. \end{aligned}$$

Define $E = \{\text{The series lasts exactly 5 games}\}$. Then

$$P(E) = \alpha + \beta = 4p(1-p)[p^3 + (1-p)^3] = 4p(1-p)[3p^2 - 3p + 1]$$