MAT1339 Notes — By Eric Hua

Introduction to Calculus and Vectors

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Introduction

Main Contents: Instantaneous rate of change as a limit, derivatives of polynomials using limits, derivatives of sums, products, the chain rule, derivatives of rational, trigonometric, exponential, logarithmic, and radical functions. Applications to finding maxima and minima and graph sketching. Concavity and points of inflection, the second derivative. Optimization in models involving polynomial, rational, and exponential functions. Vectors in two and three dimensions. Cartesian, polar and geometric forms. Algebraic operations on vectors, dot product, cross product. Applications to projections, area of parallelograms, volume of parallelepipeds. Scalar and vector parametric form of equations of lines and planes in two and three dimensions. Intersections of lines and planes. Solution of up to three equations in three unknowns by elimination or substitution. Geometric interpretation of the solutions.

Prerequisite: Ontario 4U Functions (MHF4U) or MAT1318 or equivalent. Credits for this course are in addition to the requirements of all programs of the Faculties of Science and of Engineering and of the School of Management. The courses MAT1339, Ontario 4U Calculus and Vectors (MCV4U) or any equivalent cannot be combined for credits.

Topic 1. Rate of Change (1.1, 1.2)

Understanding the nature of change and the rate at which it takes place enables us to make important predictions and decisions.

1. Average rate of change refers to the rate of change of a function over an interval. It corresponds to the slope of the secant connecting the two endpoints of the interval. Slope or Rate of Change of y with respect to x through two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{\text{the change in } y}{\text{the change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

2. Instantaneous rate of change refers to the rate of change at a specific point. It corresponds to the slope of the tangent passing through a single point, or tangent point, on the graph of a function.

3. For a given function $y = f(x)$, the instantaneous rate of change at $x = a$ is estimated by calculating the slope of a secant over a very small interval, $a \leq x \leq a + h$, where h is a very small number:
$$\frac{f(a+h) - f(a)}{(a+h) - a}.$$

Example 1 *Find a line whose graph passes through points* $(1, 2)$ *and* $(3, 6)$.

Example 2 *A line is given by* $y = -3x + 1$. *If* x *increases by 5, how does* y *change?*

Example 3 *A line is given by* $y = -3x + 1$. *If* x *increases by 5, how does* y *change?*

Example 4 *Estimate the instantaneous rate of change of the volume of a sphere with radius* r *at* $r = a$.

Topic 2. Limits and Continuity(1.3, 1.4)

Limits of Sequences

Sequence:
$$a_1, a_2, ..., a_n, ...$$

a_n is the nth term.

Definition 1 *If a sequence* a_n *arbitrarily closes to a unique number* L *as* n *goes to* ∞, *we say that* a_n *converges to* L, *which is written as:*
$$\lim_{n \to \infty} a_n = L.$$
Otherwise, we say the sequence diverges.

Example 5 *State the limit of each sequence, if it exists. If it does not exist, explain why.*

1. $\left\{\frac{(-n)^n}{n^n}\right\}$;
2. $\{(-2)^n\}$;

3. $\left\{(\frac{1}{3})^n\right\}$;

4. $\{a_n\}$: $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$;

5. $\left\{\frac{n}{n^3+1}\right\}$;

6. $\left\{\sqrt{n^2+3n-1} - \sqrt{n^2-1}\right\}$.

Limits of Functions

Function: *A function is a relationship between two variables (one is called independent variable, another is called dependent variable) such that to each value of the independent variable there corresponds exactly one value of the dependent variable.*

- Domain of the function $y = f(x)$: $D =$ The set of all values of the independent variable x for which the function is defined.

- Range of the function: $R =$ The set of all values taking on by the dependent variable y.

Definition 2 *We write*

$$\lim_{x\to a} f(x) = L$$

and say "as x approaches a, the limit of $f(x)$ is L." If L is a finite number, we say that the limit exists, otherwise, the limit does not exist.

Properties:

- $$\lim_{x\to a} P(x) = P(a), \quad P(x) \text{ is a polynomial.}$$

- $$\lim_{x\to a}(cf(x) \pm dg(x)) = cf(a) \pm dg(a), \quad \text{c, d are constants.}$$

- $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad if \ \lim_{x\to a} g(x) \neq 0.$$

- $$\lim_{x\to a}[f(x)]^n = [f(a)]^n.$$

- $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}.$$

Example 6

$$\lim_{x \to 1}(x^2 - 3) = 1^2 - 3 = -2, \quad \lim_{x \to 1} \frac{3x^4 + 8x - 2}{x - 2} = \frac{3(1)^4 + 8(1) - 2}{1 - 2} = -9.$$

Special case:

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad where\ g(a) = 0.$$

- If $f(a) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.
- If $f(a) = 0$, simplify $\frac{f(x)}{g(x)}$ first, then study the limit.

Example 7

$$\lim_{x \to 2} \frac{3x^4 + 8x - 2}{x - 2} \nexists, \quad \lim_{x \to 2} \frac{x - 2}{x - 2} = 1.$$

Example 8

$$\begin{aligned}
\lim_{x \to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2}(x + 2) = 4, \\
\lim_{h \to 0} \frac{(h + 1)^2 - 1}{h} &= \lim_{h \to 0} \frac{h(h + 2)}{h} = \lim_{h \to 0}(h + 2) = 2, \\
\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} &= \lim_{x \to 0} \frac{(\sqrt{x + 4} - 2)(\sqrt{x + 4} + 2)}{x(\sqrt{x + 4} + 2)} = \lim_{x \to 0} \frac{x}{x(\sqrt{x + 4} + 2)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x + 4} + 2} = \frac{1}{4}.
\end{aligned}$$

One-sided limit

Definition 3 *We write*

$$\lim_{x \to a^-} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the left. Similarly, We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the right.

Theorem 1

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L \ and \ \lim_{x \to a^+} f(x) = L.$$

Example 9 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0, \Rightarrow \lim_{t \to 0} H(t) \nexists.$$

Example 10 $\lim_{x \to 0} \frac{|x|}{x} \nexists.$

$\because \lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$

Example 11 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x \to 0} f(x) \nexists$ *and* $\lim_{x \to 1} f(x) = 4.$

Continuity

Definition 4 *If* $\lim_{x \to a} f(x) = f(a)$*, then* $f(x)$ *is continuous at* $x = a$*, otherwise,* $f(x)$ *is discontinuous at* $x = a$*. If* $f(x)$ *is continuous at any point on an interval, then* $f(x)$ *is continuous on the interval.*

Explore discontinuity of the function $f(x)$ at $x = a$:

- Jump Discontinuity: The one sided limits exist, but not equal.
- Infinite Discontinuity: either left side or right side limit does not exist.
- Removable Discontinuity: The one sided limits exist, and equal, but the function is undefined at $x = a$.

Example 12 *Consider* $f(x) = \frac{x^2 - 2x + 1}{x - 1}$ *at* $x = 1$.

Sol: $f(x)$ is undefined at $x=1$. But $\lim_{x\to 1} f(x)=0$. So the discontinuous point $x=1$ is **removable** if we define $f(1)=0$.

Example 13 *Determine the continuity of* $f(x)=\frac{|x|}{x}$.

Sol: $x=0$ is not removable.

Definition 5 *If* $\lim_{x\to a^-} f(x)=f(a)$, *then* $f(x)$ *is continuous from the left at* $x=a$; *if* $\lim_{x\to a^+} f(x)=f(a)$, *then* $f(x)$ *is continuous from the right at* $x=a$. *If* $f(x)$ *is continuous in* (a,b), *and left-continuous at* b, *right-continuous at* a, *then we say that* $f(x)$ *is continuous on* $[a,b]$.

Example 14 *Determine the left and right continuity at* $x=0$:

$$f(x)=\begin{cases} \frac{|x|}{x}, & x\neq 0; \\ 1, & x=0. \end{cases}.$$

Sol: continuous from right at $x=0$, discontinuous from left at $x=0$.

Theorem 2 *If* $f(x)$ *and* $g(x)$ *are continuous at* a, *then*

$$f\pm g, fg, cf (c \text{ is a constant}), \frac{f}{g} (\text{if } g(a)\neq 0)$$

are continuous.

Theorem 3 *Polynomials, rational functions, root functions are continuous in their domain.*

Example 15 *The greatest integer function:* $f(x)=[x]=$ *greatest integer less than or equal to* x *is discontinuous at any integer.*

[3.5]=3, [-3.5]=-4, [5]=5.

Example 16 *Find* k *such that* $f(x)=\begin{cases} x^3+kx^2-5x, & x>2; \\ \frac{x}{x-3}, & x\leq 2 \end{cases}$ *is continuous at any* x.

Topic 3. Derivatives(1.5, 2.1-2.6)

1.5 First principles definition of the derivative

Definition 6 *(First principles definition of the derivative): The derivative of the function $y = f(x)$ is the function $f'(x)$:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Meaning: $f'(a) =$

- instantaneous rate of change of $f(x)$ at a, or
- the slope of the tangent line to the curve at a.

Example 17 *Let $f(x) = x^2$. Calculate $f'(5)$.*

Example 18 *The volume of a sphere of radius r is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ by definition. What's the meaning of this derivative?

Example 19 *Let $f(x) = \sqrt{x-3}$. Find $f'(x)$ and state the domains of f and f'.*

Example 20 *Let $f(x) = \frac{x}{x-3}$. Find $f'(x)$ and state the domains of f and f'.*

Example 21 *Find $f'(x)$ from the graph of f.*

Definition 7 *The function f is differentiable at a if $f'(a)$ exists. It is differentiable on an interval if $f'(a)$ exists for any a on the interval.*

Example 22 *$f(x) = |x|$ is not differentiable at $x = 0$.*

Theorem 4 *If a function is differentiable at $x = c$, then the function is continuous at $x = c$.*

2.1 Some Differentiation Rules

- Constant rule: If $f(x) = c$, then $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.
- Constant multiple rule: $[cf(x)]' = cf'(x)$.
- Sum rule and difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$.
- Power Rule: If $f(x) = x^n$, n is any real number. Then $f'(x) = nx^{n-1}$.

Example 23 *Let* $f(x) = 5$. *Then* $f'(x) = 0$.
Let $f(x) = x^5$. *Then* $f'(x) = 5x^4$.
Let $f(x) = \frac{1}{x^5}$. *Then* $f'(x) = (x^{-5})' = -5x^{-6}$.
Let $f(x) = \frac{1}{\sqrt{x}}$. *Then* $f'(x) = (x^{-0.5})' = -0.5x^{-1.5}$.
Let $f(x) = \frac{2}{\sqrt{x}}$. *Then* $f'(x) = 2\left(\frac{1}{\sqrt{x}}\right)' = -x^{-1.5}$.

Example 24 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Find the equation of the tangent line at* $(1, -6)$.

Example 25 *Find the equation of the tangent line to* $f(x) = 2\sqrt{x} - 3$ *at* $(4, 1)$.

Sol: $f'(x) = x^{-1/2} \Rightarrow f'(4) = 1/2 \Rightarrow y = 1/2x - 1$.

Example 26 *Find the equation of the line(s) that pass through the point* $P(2, 9)$ *and are tangent to* $f(x) = -x^2 + 2x$. *Sketch the graph.*

Sol: $f'(x) = -2x + 2$. Let $(a, f(a))$ be a point on the curve whose tangent line goes through $P(2, 9)$. Then $m = -2a + 2 \Rightarrow$

$$\frac{f(a) - 9}{a - 2} = -2a + 2, \Rightarrow a = -1, 5.$$

When $a = -1$, $m = 4 \Rightarrow y = 4x + 1$;
When $a = 5$, $m = -8 \Rightarrow y = -8x + 25$.

2.2 The product rule

- Product rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

Example 27 *Let* $f(x) = (\sqrt{x} + x^2)(x^3 + x)$*. Calculate* $f'(4)$*.*

Sol: By the product rule,

$$f'(x) = (\frac{1}{2\sqrt{x}} + 2x)(x^3 + x) + (\sqrt{x} + x^2)(3x^2 + 1) \Rightarrow f'(4) = (\frac{1}{4} + 8)(68) + 18(49).$$

Example 28 *Let* $f(x) = (x^3 + 4x^2)(x^5 + x + 1)$*. Calculate* $f'(1)$ *and the tangent at* $(1, 15)$*.*

2.5 The quotient rule

- Quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 29 *Let* $f(x) = \frac{\sqrt{x}+x^2}{x^3+x}$*. Calculate* $f'(4)$*.*

Sol: Use the quotient rule.

Example 30 *. Let* $f(x) = \frac{x^3+4x^2}{x^5+x+1}$*. Calculate* $f'(1)$*.*

Sol: Use the quotient rule.

Example 31 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*

a) parallel to $y = 3x$*?*

b) perpendicular to $y = -0.5x$*?*

Solution: By quotient rule,

$$y' = \frac{(x^2-4)'(x+1)-(x^2-4)(x+1)'}{(x+1)^2} = \frac{2x(x+1)-(x^2-4)1}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2}.$$

a) Let $y'=3 \Rightarrow \frac{x^2+2x+4}{(x+1)^2}=3 \Rightarrow 2x^2+4x-1=0 \Rightarrow x=-1\pm\frac{\sqrt{6}}{2}$.

b) $y'=-\frac{1}{-0.5}=2 \Rightarrow \frac{x^2+2x+4}{(x+1)^2}=2 \Rightarrow x^2+2x-2=0 \Rightarrow x=-1\pm\frac{\sqrt{3}}{2}$.

2.4 The chain rule

- Chain Rule:

$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv}\cdot\frac{dg(x)}{dx}, v=g(x), \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

- General Power Rule:

$$[u(x)^n]' = nu^{n-1}u'(x).$$

Example 32 *Let $f(x)=(x^2-x-1)^{100}$. Calculate $f'(x)$.*

Sol: $f'(x)=100(x^2-x-1)^{99}(x^2-x-1)'=100(x^2-x-1)^{99}(2x-1)$.

Example 33 *Let $h(x)=g(f(x))$, where $f'(2)=3$, $f(2)=4$, $g'(3)=-5$, $g(4)=8$, $g'(4)=7$. Find $h'(2)$.*

Solution: $h'(x)=g'(f(x))f'(x) \Rightarrow h'(2)=g'(f(2))f'(2)=g'(4)(3)=7(3)=21$.

Example 34 *Let $y=\sqrt{x+\sqrt{x^2+x}}$. Calculate y'.*

Solution:

$$y' = \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x^2+x}}}(x+\sqrt{x^2+x})'$$

$$= \frac{1}{2\sqrt{x+\sqrt{x^2+x}}}\left(1+\frac{1}{2}\frac{1}{\sqrt{x^2+x}}(x^2+x)'\right) = \frac{1}{2\sqrt{x+\sqrt{x^2+x}}}\left(1+\frac{2x+1}{2\sqrt{x^2+x}}\right)$$

2.3 Velocity, Acceleration, and Second Derivatives

Let $y = f(x)$. Then

$$y''(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx}\left(\frac{dy^{(n-1)}}{dx}\right).$$

Example 35 *Let $f(x) = 4x^3+6x^2-23x+7$. Then $f'(x) = 12x^2+12x-23$, $f''(x) = 24x+12$, $f'''(x) = 24$ and $f^{(4)}(x) = 0$.*

Example 36 *Let $f(x) = (x^2 - x - 1)^{100}$. Calculate $f''(x)$.*

Example 37

$$(x^n)^{(n)} = n!, \quad (\frac{1}{x})^{(n)} = (-1)^n n! x^{-n-1}.$$

Velocity and Acceleration

Definition 8 *Let $s = f(t)$ be the position function. The s is called the displacement at time t. The velocity at t is defined as*

$$v(t) = f'(t).$$

The acceleration at t is defined as

$$a(t) = f''(t) = v'(t).$$

Example 38 *The position of a particle is given by*

$$s = t^3 - 15t^2 + 63t, \quad t \geq 0$$

where s is measured in meters and t in seconds.

a) What is the initial position? initial velocity? initial acceleration?

b) Find the velocity after 1s and 4s.

c) When is the particle at rest?

d) When is the particle moving in the positive direction?

e) When is the acceleration 0?

f) Determine when the particle is speeding up and slowing down.

Solution:

$$s = t^3 - 15t^2 + 63t, \Rightarrow s'(t) = 3t^2 - 30t + 63, \Rightarrow s''(t) = 6t - 30.$$

a) $s(0) = 0, v(0) = s'(0) = 63, a(0) = s''(0) = -30.$

b) $v(1) = s'(1) = 36$, $v(4) = s'(4) = -9$.
c) $s'(t) = 3t^2 - 30t + 63 = 0, \Rightarrow t = 3, 7$.
d) $s'(t) = 3t^2 - 30t + 63 > 0, \Rightarrow 0 < t < 3$, or $t > 7$.
e) $s'' = 0 \Rightarrow t = 5$.
f) Speeding up at time t, if $v(t)a(t) > 0$, i.e., $(3t^2 - 30t + 63)(6t - 30) > 0$, i.e.,

$$18(t-3)(t-5)(t-7) > 0, \Rightarrow 3 < t < 5, or, t > 7.$$

Slowing down at time t, if $v(t)a(t) < 0$, i.e., $(3t^2 - 30t + 63)(6t - 30) < 0$, i.e.,

$$18(t-3)(t-5)(t-7) < 0, \Rightarrow 5 < t < 7, or, t < 3.$$

2.6 Rate of Change Problems

This section will focus on applying derivatives to solve problems involving rates of change in Business and Economics.

Functions Pertaining to Business: Let x be the number of units of a product or service sold at a price $p(x)$ per unit.

- The demand function is $p(x)$
- The revenue function is $R(x) = xp(x)$
- The cost function , C(x), is the total cost of producing x units of a product or service.
- The profit function , P(x), is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$.

Derivatives of Business Functions: Economists use the word marginal to indicate the derivative of a business function.

- $C'(x)$ is the marginal cost function and refers to the instantaneous rate of change of total cost with respect to the number of items produced.
- $R'(x)$ is the marginal revenue function and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.
- $P'(x)$ is the marginal profit function and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

Example 39 *A store has been selling Häagen-Dazs bar at the price of $3.00 per bar and, at this price, the store can sell 60 bars per day. If the store raises its price, it will sell 2 fewer bars per day for each $0.4 increase in price. Assume that x bars cost the store* $0.1x^2 + 0.4$.

a. Find the demand equation (the price p as a function of quantity x)
b. Find the marginal cost.
c. Find the revenue function and the marginal revenue.
d. Find the profit function and the marginal profit.

Topic 4. Curve Sketching(3.1-3.6)

3.1 Increasing and Decreasing Functions

Definition 9 *$y = f(x)$ is increasing on an interval I if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$; $y = f(x)$ is decreasing on an interval I if $f(x_1) \geq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$.*

INCREASING/DECREASING TEST (I/D TEST):

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval, then f is a constant on that interval.

Definition 10 *Critical number: A point p in the domain such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number, $(p, f(p))$ is a critical point, $f(p)$ is a critical value.*

Example 40 *Let $f(x) = x^4 - 5x^3 + x^2 + 21x - 18$. Find all critical numbers.*

Example 41 *Let $f(x) = x^3 - 3x^2$. State all the intervals of increase and decrease.*

Solution:

(a) $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Let $f'(x) = 0$. We have $3x(x - 2) = 0$, which gives $x = 0, 2$.

(b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	$+$	-	$+$
$f(x)$	increase	decrease	increase

Therefore,

The intervals of increase: $-\infty < x < 0$, $2 < x < \infty$.

The intervals of decrease: $0 < x < 2$.

Example 42 *Find the critical numbers of* $f(x) = x^{3/5}(4 - x)$.

Sol: $f'(x) = \frac{12-8x}{5x^{2/5}}$. $f'(x) = 0 \Rightarrow x = 1.5$; $f'(x)$ undefined at $x = 0$. The critical numbers are 1.5 and 0.

Example 43 *Find the critical numbers of* $f(x) = |x|$.

Sol:

$$f'(x) = \begin{cases} 1, & x > 0; \\ -1, & x < 0. \end{cases}$$

$f'(x)$ does not exit at $x = 0$.

3.2 Maxima and Minima

- Local (relative) extrema: $f(x)$ has a local minimum at p if $f(p) \leq f(x)$ for points x near p; $f(x)$ has a local maximum at p if $f(p) \geq f(x)$ for points x near p.
- Absolute Maximum and Minimum: $f(x)$ has an absolute Maximum at p if $f(p) \geq f(x)$ for all x in the domain; $f(x)$ has Absolute Minimum at p if $f(p) \leq f(x)$ for all x in the domain.

First Derivative Test: Let p be a critical number. If f' changes from - to + at p, then f has a local minimum at p; If f' changes from + to - at p, then f has a local maximum at p.

Example 44 *Let* $f(x) = x^3 - 3x^2$. *Find the local minimum points and all the local maximum points.*

Solution:

(a) $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Let $f'(x) = 0$. We have $3x(x - 2) = 0$, which gives $x = 0, 2$.

(b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

(c) Note that at $x = 0$, $f'(x)$ changes from + to -; at $x = 2$, $f'(x)$ changes from - to +. By the First Derivative Test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.

FERMAT THEOREM: If f has a relative maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.
CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ on a closed interval $[a, b]$:

1. Find all the critical numbers, e.g., x_1, ..., x_n.
2. global minimum = $\min\{f(x_1), ..., f(x_n), f(a), f(b)\}$;

   global maximum = $\max\{f(x_1), ..., f(x_n), f(a), f(b)\}$.

Example 45 *Find the absolute maximum and minimum of the function*

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Sol: Step 1) $f'(x) = 6x^2 - 6x - 12$, $f'(x) = 0 \Rightarrow x = -1, 2$, $f'(x)$ is defined anywhere. Hence $x = -1$ is the only one critical number in $(-2, 0)$.

Step 2) global minimum = $\min\{f(-2), f(-1), f(0)\} = \min\{3, 14, 7\} = 3$;
global maximum = $\max\{f(-2), f(-1), f(0)\} = \max\{3, 14, 7\} = 14$.

3.3 Concavity and the Second Derivative Test

Definition 11 *(CONCAVITY)If the graph of f lies above all of its tangents on an interval I (f' is increasing on I), it is called concave upward on I. If the graph of f lies below all of its tangents on I (f' is decreasing on I), it is called concave downward on I.*

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 12 *Point of inflection: If $f(x)$ changes concavity at p, then p is an inflection point, and $f''(p) = 0$ or undefined.*

Example 46 *Let $f(x) = x^3 - 3x^2$. State intervals of concavity and find all the points of inflection.*

Solution: $f''(x) = 6x - 6 = 0 \Rightarrow x = 1$.

x	$-\infty < x < 1$	$1 < x < \infty$
$f''(x)$	-	+
$f(x)$	concave down	concave up

Concave up: $1 < x < \infty$; Concave down: $-\infty < x < 1$.

Since $f(x)$ changes concavity at $x = 1$, $x = 1$ is a point of inflection.

Second Derivative Test: Let p be a critical number.

- If $f''(p) > 0$, then f has a local minimum at p;
- If $f''(p) < 0$, then f has a local maximum at p;
- If $f''(p) = 0$, then nothing.

Example 47 *Let $f(x) = x^3 - 3x^2 - 80$. Find all the local minimum points and all the local maximum points by Second Derivative Test.*

Sol: From $f'(x) = 3x^2 - 6x$ we get $f''(x) = 6x - 6$. $f'(x) = 0 \Rightarrow x = 0, 2$. Note that $f''(0) = -6 < 0$ $f''(2) = 6 > 0$ By the Second Derivative Test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.

3.4 Rational Functions

Definition 13 *The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x) = \pm\infty, \lim_{x\to a^+} f(x) = \pm\infty, \lim_{x\to a} f(x) = \pm\infty.$$

Example 48 $\lim_{x\to 0} \frac{1}{x^2} = \infty$, $\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$.

Example 49 *Verify VA from graph, sketch graph.*

Definition 14 *If $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a HA (Horizontal Asymptote).*

Example 50 *Find HA:*

$$f(x) = \frac{2x^2 - 1}{x^2}, \quad \frac{P(x)}{Q(x)}.$$

Example 51 *Verify HA from graph, sketch graph.*

3.5 Curve Sketching

1. EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.

2. ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.

To sketch a polynomial, follow these steps:

- Step 1 Determine the domain of the function.
- Step 2 Determine the intercepts of the function.
- Step 3 Determine and classify the critical numbers of the function.
- Step 4 Determine the points of inflection.
- Step 5 Determine the intervals of increase and decrease and the intervals of concavity of the function.
- Step 6 Sketch the function.

Example 52 *Analyze and sketch the following function:*

$$g(x) = 3x^4 + 2x^3 - 15x^2 + 12x - 2.$$

To sketch a rational function, follow these steps:

- Step 1 Determine the domain of the function.
- Step 2 Determine the intercepts of the function.
- Step 3 Determine vertical and horizontal asymptotes.
- Step 4 Determine and classify the critical numbers of the function.
- Step 5 Determine the points of inflection.
- Step 6 Determine the intervals of increase and decrease and the intervals of concavity of the function.
- Step 7 Sketch the function.

Example 53 *Sketch the graph*

$$f(x) = \frac{x}{x^2 - 1}.$$

Solution:

A. The domain of the function: $x \neq \pm 1$.

B. x-intercept: $y = 0 \Rightarrow x = 0$; y-intercept: $x = 0 \Rightarrow y = 0$.

C. $\lim_{x\to\pm\infty} f(x) = 0$, $y = 0$ is HA; VA: let $x^2 - 1 = 0$, we have $x = 1$ and $x = -1$.

D. $f'(x) = \frac{-1-x^2}{(x^2-1)^2}, \Rightarrow$ No critical number.

E. Note that $f'(x) < 0$ for any x, so f is decreasing in the domain.

F. $f'' = \frac{2x(x^2+3)}{(x^2-1)^3}$.

$$f''(x) = 0, \Rightarrow x = 0.$$

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

3.6 Optimization Problems

GUIDELINES FOR SOLVING MAX./MIN. PROBLEMS

1. If possible, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.

2. Define variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

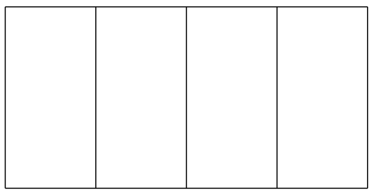
3. Write down all equations which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize. Experience will show you that MOST optimization problems will begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. Some problems may have NO constraint equation. Some problems may have two or more constraint equations.

4. Change your equations to a function of only one variable. Then differentiate the function.

5. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

Example 54 *Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.*

Example 55 *Build a rectangular garden with three parallel partitions using 500 feet of fencing (See the graph). What dimensions will maximize the total area of the garden?*



Example 56 *Suppose that a rectangle is to be inscribed in a upper semicircle of radius 2. What is the largest area the rectangle can have?*

Example 57 *Find the point(s)* (x, y) *on the graph of* $y = \sqrt{x}$ *nearest the point* $(4, 0)$.

Example 58 *A container has volume* $24000\pi cm^3$. *The material for the cover costs 0.004\$ per* cm^2, *other parts cost 0.002\$ per* cm^2. *Find the dimensions that minimize the total cost.*

Example 59 *A store has been selling Häagen-Dazs bar at the price of \$3.00 per bar and, at this price, the store can sell 60 bars per day. If the store raises its price, it will sell 2 fewer bars per day for each \$0.4 increase in price. Assume that* x *bars cost the store* $0.1x^2 + 0.4$.

a. *Find the demand equation (the price* p *as a function of quantity* x*)*
b. *Find the minimum average cost.*
c. *Find the maximum revenue.*
d. *Find the maximum profit.*

Topic 5. Derivatives of Sinusoidal Functions(4.1-4.4)

4.1 Sinusoidal functions

Radian $\leftrightarrows$ Degree: t degree $= \frac{t}{180}\pi$.
Consider a right triangle:

$$\sin t = \frac{opposite}{hypothenuse}, \quad \cos t = \frac{adjacent}{hypothenuse}.$$

Basic relations:

$$\begin{array}{l|l} \sin(\theta + \frac{\pi}{2}) = +\cos\theta & \sin(\theta+\pi) = -\sin\theta \\ \cos(\theta + \frac{\pi}{2}) = -\sin\theta & \cos(\theta+\pi) = -\cos\theta \end{array}$$

Pythagorean trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

Special values:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

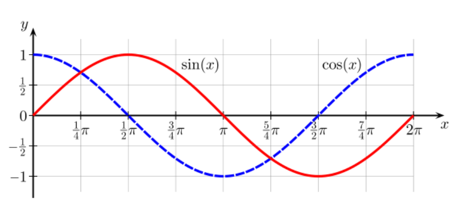
Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Half-angle formula.

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Periods: $\sin x$ and $\cos x$ have period 2π, $\tan x$ and $\cot x$ have period π.
Graphs.



Example 60 *Find all values of x in the interval $[0, 2\pi]$ such that* $\sin^2 x - 3\cos^2 x = 0$.

Solution: $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$.

Example 61 *Find* $\cos x$ *where* $x \in [\frac{\pi}{2}, 2\pi]$ *such that* $\sin x = 0.8$.

Solution: $\cos x = -0.6$

Example 62 *Evaluate* $\sin \frac{\pi}{12}$.

Solution: $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12}$.

Special inequalities:

$$-|x| \leq \sin x \leq |x|, \quad -|x| \leq 1 - \cos x \leq |x|.$$

4.2-4.4 Derivatives of the Sine and Cosine Functions

Recall the result:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Derivative of Sine and Cosine Functions:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

Example 63 *Let* $y = \sin(x)$*, calculate* $y''(x)$*.*

Example 64 *Find the equation of the tangent line to the curve* $x^3 \sin(5x^2 - 5)$ *at* $(1, 0)$*.*

Example 65 *Find* $f'(x)$*. If*

$$f(x) = \sin x^2, \quad \sin^2 x, \quad x^3 \cos(5x^2), \quad \frac{1 + \cos x}{1 + \sin x}, \quad \sin(\cos(3x^2 - 1)).$$

Example 66 *Find the points on the curve* $f(x) = -2\cos x + \sin(2x)$ *where the tangent line is horizontal.*

Example 67 *A power supply delivers a voltage signal that consists of an alternating current (AC) component and a direct current (DC) component.The voltage, in volts, at time t (in seconds) is given by the function* $V(t) = 5\sin(2t) + 9$*.*

(a) Find the maximum and minimum voltages. At which times do these values occur?

(b) Determine the period, T, in seconds, frequency, f, in hertz, and amplitude, A, in volts, for this signal.

Solution:

(a) $max = 5 + 9 = 14$, at $2t = 2k\pi + \frac{\pi}{2}$, $k = 0, 1, 2, ...$;
$min = -5 + 9 = 4$, at $2t = 2k\pi + \frac{3\pi}{2}$, $k = 0, 1, 2, ...$

(b) $T = \frac{2\pi}{2}$, $f = \frac{1}{T}$, $A = \frac{V_{max} - V_{min}}{2} = 5$.

Topic 6 Exponential and Logarithmic Functions (5.1-5.5)

5.1 and 5.5 Exponential Functions

An exponential function is defined as:

$$y = f(x) = a^x, \quad a > 0.$$

- Domain: x can be any real number.
- Exponential growth: $a > 1$; $\lim_{x\to-\infty} a^x = 0$, $\lim_{x\to\infty} a^x = \infty$.
- Exponential decay: $0 < a < 1$; $\lim_{x\to-\infty} a^x = \infty$, $\lim_{x\to\infty} a^x = 0$.

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Example 68 *Sketch the graph of* $y = 2^x + 5$.

Solution: HA: $\lim_{x\to-\infty} y = 5$.

Example 69 *Sketch the graph of* $y = 2^{-x} + 5$.

Solution: HA: $\lim_{x\to\infty} y = 5$.

Natural exponential function is defined as:

$$y = f(x) = e^x, \quad a > 0,$$

where $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = \lim_{x\to 0}(1 + x)^{1/x} \doteq 2.71828....$

Application in compounded interest. Let A represent the amount of money after a certain amount of time, P represent the principle or the amount of money you start with (Present value), r represent the interest rate, t represent the amount of time in years, n represent the number of times per year. Then

- Compounded n times per year:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

- Compounded continuously:

$$A = Pe^{rt}.$$

Example 70 *Suppose $12000 is put into an account that pays 11% annually. How much will be in the account after 25 years?*

a) compounded continuously.

b) compounded quarterly.

Solution: a)

$$A = Pe^{rt} = 12000e^{0.11(25)} = 187711.58.$$

b)

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 12000\left(1 + \frac{0.11}{4}\right)^{4(25)} 180869.07$$

Application in population. Let $P(t)$ be population at the time t, P_0 be the initial quantity.

$$P(t) = P_0 a^t, \quad a > 0, a \neq 1.$$

where

- Exponential growth: $a > 1$;
- Exponential decay: $0 < a < 1$.
- $a = P(t+1)/P(t)$.

Special cases:

- Half-life (exponential decay): The time required for the quantity to be reduced to half. Let H be the half-life, then

$$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$

- Doubling-time (exponential growth): The time required for the quantity to be doubled. Let D be the doubling time, then

$$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$

Example 71 *A bacterial culture starts with 500 bacteria and doubles in size every hour.*

a) How many are there after t hours?

b) How many are there after 10 minutes?

Solution: a) Let $P(t)$ be the number after t hours. Then $P(0) = 500$, $P(t+1) = 2P(t)$. $D = 1$.

$$P(t) = (500)2^{t/1} = (500)2^t.$$

b) $P(10/60) = (500)2^{10/60} = (500)2^{1/6}$.

5.2 The Natural Logarithm

$$y = a^x \quad \underrightarrow{inversefunction} \quad y = \log_a x,$$
$$y = e^x \quad \underrightarrow{inversefunction} \quad y = \log_e x = \ln x,$$
$$y = 10^x \quad \underrightarrow{inversefunction} \quad y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ is called logarithmic function with the base a. Domain $= \{x > 0\}$. $y = \ln x$ is called natural logarithmic function.

Properties: Let $B, C > 0$. Then

1. $\ln(BC) = \ln B + \ln C$,
2. $\ln(\frac{B}{C}) = \ln B - \ln C$,
3. $\ln(B^n) = n \ln B$,
4. $\ln(e^x) = x$, $\ln e = 1$,
5. $e^{\ln B} = B$,
6. $\log_e 1 = 0$.

Example 72 *Convert a^x to base e.*

$$a^x = e^{x \ln a}.$$

Example 73 *Solve for x:*

$$3^{2x-1} = 4, \quad \ln x + \ln(x-8) = \ln 9.$$

Example 74 *A bacterial culture starts with 500 bacteria and triples in size every 4 hours.*

a) How many are there after t hours?

b) How long will the population be 600?

Solution: a) Let $P(t)$ be the number after t hours. Then $P(0) = 500$.

$$P(t) = (500)3^{t/4}.$$

b) $600 = (500)3^{t/4}, \Rightarrow 1.2 = 3^{t/4}, \Rightarrow \ln 1.2 = \frac{t}{4}\ln 3, \Rightarrow t = \frac{4\ln 1.2}{\ln 3}$.

5.3-5.4 Derivatives of Exponential and Logarithmic Functions

Derivative of exponential function:

$$(e^x)' = e^x, \quad (e^{u(x)})' = e^{u(x)}u'(x).$$

Example 75

$$(e^{4x})' = 4e^{4x}.$$

Example 76 *Show that $e^x \geq 1 + x$ for $x \geq 0$.*

Proof. Let $f(x) = e^x - (1 + x)$. Then $f'(x) = e^x - 1 \geq 0$ when $x \geq 0$. Thus $f(x)$ is increasing when $x \geq 0$. Note that $f(0) = 0$, so $f(x) \geq 0$ for $x \geq 0$.

Example 77 *At what point(s) on the curve $y = x^2e^x - 3e^x$ is the tangent line horizontal?*

Solution: $y' = (2x + x^2 - 3)e^x$. Let $y' = 0$, we imply that $x = 1, -3$.

Example 78 *The catenary is the theoretical shape of a hanging flexible chain or cable when supported at its ends and acted upon by a uniform gravitational force (its own weight) and in equilibrium. Let*

$$y = 30(e^{x/60} + e^{-x/60}), \quad -30 \leq x \leq 30.$$

Find the lowest point.

Solution. $y' = \frac{1}{2}(e^{x/60} - e^{-x/60})$. From $y' = 0$, $x = 0$.

x	$x < 0$	$x > 0$
y'	-	+

By the first-derivative test, the minimum point is $(0, 60)$.

Derivative of logarithmic functions:

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a},$$

$$(a^x)' = a^x(\ln a), \quad [a^{u(x)}]' = a^{u(x)}(\ln a)u'(x).$$

  Change of base:

$$\log_a b = \frac{\ln b}{\ln a}.$$

Example 79 *Differentiate* $\ln(x^2 + 1)$.

Example 80 *Differentiate* $y = \ln \frac{x^2+x+5}{(x+1)^2}$.

Topic 7 Geometric Vectors (6.1-6.5)

6.1 Introduction to Vectors

A scalar is a quantity that describes magnitude or size only, e.g., distance, speed, mass.
A vector is a quantity that has both magnitude and direction, e.g., velocity, force.

- The magnitude of vector $\overrightarrow{AB}$ or $\vec{v}$ is written as $|\overrightarrow{AB}|$ or $|\vec{v}|$.

- A vectors direction can be expressed using several different methods.

  1. In the diagram of $\overrightarrow{AB}$, it is expressed as an angle, moving counterclockwise with respect to a horizontal line.

  2. In navigation, vector directions are expressed as bearings. (a) A (true) bearing (or azimuth bearing) is a compass measurement where the angle is measured from north in a clockwise direction. True bearings are expressed as three-digit numbers, including leading zeros. For example, a bearing of 040° is an angle of 40° in a clockwise direction from due north. (b) Quadrant bearing , which is a measurement between 0° and 90° east or west of the north-south line. The quadrant bearing N23°W is read as 23° west of north, whereas S20°E is read as 20° east of south. All quadrant bearings are referenced from north or south, not from west or east.

A vector can be represented in several ways:

- In words, for example, a car travelling at 80 km/h to the west.

- In a diagram, as a geometric vector , which is a representation of a vector using an arrow diagram, or directed line segment, that shows both magnitude (or size) and direction.

- In symbols, using the endpoints of the arrow: $\overrightarrow{AB}$. Point A is the starting or initial point of the vector (also known as the tail). Point B is the end or terminal point of the vector (also known as the tip or head).

- In symbols, using a single letter: $\vec{v}$.

Relations of vectors:

- Parallel vectors have the same or opposite direction, but not necessarily the same magnitude.
- Equivalent vectors have the same magnitude and the same direction. The location of the vectors does not matter.
- Opposite vectors have the same magnitude but opposite direction.

6.2-6.4 Addition, Scalar multiplication, and Linear combination of Vectors

Resultant is a single vector resulting from several vectors.

Vector Addition: (a) head-to-tail (or triangle) method. (b) tail-to-tail (or parallelogram) method.

Scalar Multiplication: $k\vec{v}$, which has the magnitude $|k||\vec{v}|$ and the same or opposite direction with $\vec{v}$.

Example 81 *In an orienteering race, you walk 100 m due east and then walk N70°E for 60 m. How far are you from your starting position, and at what bearing?*

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $(cd)\vec{u} = c(d\vec{u})$
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

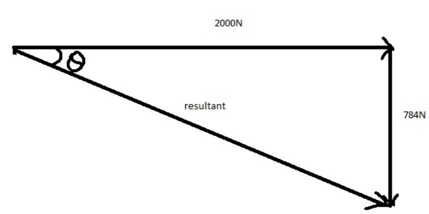
- $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$
- $1\vec{u} = \vec{u}$
- $(-1)\vec{u} = -\vec{u}$
- $0\vec{u} = \vec{0}$
- $\vec{u}//\vec{v} \Leftrightarrow \vec{v} = c\vec{u}$

An equilibrant vector is equal in magnitude but opposite in direction to the resultant vector. If the equilibrant is added to a given system of vectors, the sum of all vectors, including the equilibrant, is $\vec{0}$.

Example 82 *A clown with mass 80 kg is shot out of a cannon with a horizontal force of 2000 N. The vertical force is the acceleration due to gravity, which is 784 N ($9.8m/s^2 \times$ the mass of the clown).*

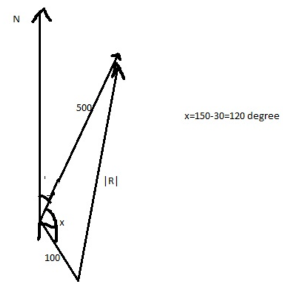
a) Find the magnitude and direction of the resultant force on the clown.

b) Find the magnitude and direction of the equilibrant force on the clown.



Heading is the direction in which a vessel is steered.
Heading velocity = air velocity.
Ground velocity is the velocity relative to the ground. It is the resultant (or bearing velocity).

Example 83 *An airplane is flying at an airspeed of 500 km/h, on a heading of 030°. A 100-km/h wind is blowing from a bearing of 150°. Determine the ground velocity of the airplane and the direction of flight.*



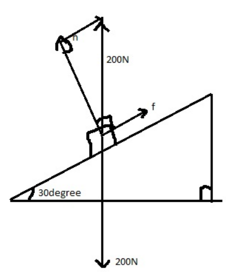
6.5 Resolution of Vectors Into Rectangular Components

Given a vector $\vec{u}$, the rectangular (or perpendicular) components are two vectors $\vec{u_1}$ and $\vec{u_2}$ such that

$$\vec{u} = \vec{u_1} + \vec{u_2} \quad and \quad \vec{u_1} \perp \vec{u_2}.$$

In this case, we say that $\vec{u}$ is resolved into $\vec{u_1}$ and $\vec{u_2}$.

Example 84 *A box weighing 200 N is resting on a ramp that is inclined at an angle of* 30°*. Resolve the weight into rectangular components that keep the box at rest.*



Topic 8 Cartesian Vectors (7.1-7.6)

7.1 Cartesian Vectors in $\mathbb{R}^2$

Algebraic representation of vectors:

- Cartesian vectors: $\vec{u} = [u_1, u_2]$. Zero vector $\vec{0} = [0, 0]$. The unit vectors $\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$. If a Cartesian vector $\vec{u}$ is translated so that its tail is at the origin, (0, 0), and its tip is at the point (u_1, u_2), the translated vector is called the position vector. Any Cartesian vector $\vec{u} = [u_1, u_2]$ can be written as the sum of its vertical and horizontal vector components,

$$\vec{u} = [u_1, 0] + [0, u_2] = u_1\vec{i} + u_2\vec{j}.$$

  The Cartesian vector between two points, $P1(x_1, y_1)$ and $P_2(x_2, y_2)$, is $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1]$. A geometric vector $\vec{u}$ can be written in Cartesian form as

$$\vec{u} = [\vec{u}\cos\theta, \vec{u}\sin\theta],$$

  where θ is the angle between $\vec{u}$ and the positive x-axis.

- Magnitude: $|[a, b]| = \sqrt{a^2 + b^2}$. Unit vector: $|\vec{u}| = 1$.

- Addition: Let $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$, then $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2]$.

- Subtraction: Let $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$, then $\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2]$.

- Scalar multiplication: Let $\vec{u} = [u_1, u_2]$, c be a scalar, then $c\vec{u} = [cu_1, cu_2]$.

Example 85 *1. Write a force of 100N at $\frac{\pi}{6}$ to the horizontal in the cartesian plane.*

2. Find two unit vectors collinear with $\vec{u} = [3, 4]$

7.2-7.3 Dot product and Applications

- Dot product: Let $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$.

- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \leq \theta \leq \pi$, then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$.

- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.

Example 86 *Let $\vec{u} = [3, -4]$, $\vec{v} = [5, 12]$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Properties: Let c be a scalar.

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{w} \cdot (\vec{u} + \vec{v}) = \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v}$
- $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
- $\vec{u} \cdot \vec{0} = 0$
- $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.

Example 87 *Determine the values of k so that $\vec{u} = [3, k]$, $\vec{v} = [2k, k]$ are perpendicular.*

Vector Projections: The projection of $\vec{v}$ on $\vec{u}$ is defined as

$$\text{proj}_u \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}.$$

Example 88 *Let $\vec{u} = [3, -4]$, $\vec{v} = [5, 12]$. Find $proj_u \vec{v}$ and $proj_v \vec{u}$.*

7.4 Cartesian Vectors in $\mathbb{R}^3$

Algebraic representation of vectors:

- Cartesian vectors: $\vec{u} = [u_1, u_2, u_3]$. Zero vector $\vec{0} = [0, 0, 0]$. The unit vectors $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, $\vec{k} = [0, 0, 1]$. If a Cartesian vector $\vec{u}$ is translated so that its tail is at the origin, (0,0,0), and its tip is at the point (u_1, u_2, u_3), the translated vector is called the position vector. Any Cartesian vector $\vec{u} = [u_1, u_2, u_3]$ can be written as

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}.$$

The Cartesian vector between two points, $P1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, is $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$.

- Magnitude: $||[a, b, c]|| = \sqrt{a^2 + b^2 + c^2}$. Unit vector: $|\vec{u}| = 1$.
- Addition: Let $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$, then $\vec{u} + \vec{v} = [u_1 + v_1, u_2 + v_2, u_3 + v_3]$.
- Subtraction: Let $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$, then $\vec{u} - \vec{v} = [u_1 - v_1, u_2 - v_2, u_3 - v_3]$.
- Scalar multiplication: Let $\vec{u} = [u_1, u_2, u_3]$ c be a scalar, then $c\vec{u} = [cu_1, cu_2, cu_3]$.
- Dot product: Let $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$, then $\vec{u} \cdot \vec{v}) = u_1v_1 + u_2v_2 + u_3v_3$.
- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \le \theta \le \pi$, then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$.
- The projection of $\vec{v}$ on $\vec{u}$ is defined as

$$\text{proj}_u\vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}.$$

Example 89 *Let $\vec{u} = [2, -2, 1]$, $\vec{v} = [2, 1, 2]$. Find proj$_u\vec{v}$ and proj$_v\vec{u}$.*

Example 90 *Determine if the vectors $\vec{u} = [-2, 6, 4]$ and $\vec{v} = [-3, 9, 6]$ are collinear.*

Example 91 *Find a vector that is orthogonal to [-3, 4, 5].*

Example 92 *A force with units in newtons and defined by $\vec{F} = [300, 700, 500]$ acts on an object with displacement, in metres, defined by $\vec{d} = [3, 1, 12]$.*

a) Determine the work done in the direction of travel.

b) Determine the work done against gravity, which is a force in the direction of the negative z-axis.

7.5-7.6 Cross product and Applications

- Cross product: Let $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$, then
  $\vec{u} \times \vec{v} = [u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$.

Example 93 *Find a vector that is orthogonal to both* $\vec{u} = [1, 2, -1]$, $\vec{v} = [0, 2, 3]$.

Properties: Let c be a scalar.

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$, $\vec{u} \times \vec{u} = \vec{0}$
- Orthogonal: $(\vec{u} \times \vec{v}) \perp \vec{u}$, $(\vec{u} \times \vec{v}) \perp \vec{v}$.
- $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$

Applications:

- The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.
- The volume of the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$: $|\vec{u} \cdot (\vec{v} \times \vec{w})|$.
- Torque, represented by the Greek letter tau (τ), is a measure of the force acting on an object that causes it to rotate:
  $$\vec{\tau} = \vec{r} \times \vec{F},$$
  where $\vec{F}$ is the force acting on the object, $\vec{r}$ represents the arm.

Example 94 *Find the area of the parallelogram determined by* $\vec{u} = [1, 2, -1]$, $\vec{v} = [0, 2, 3]$.

Example 95 *Find the area of the triangle with vertices* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Example 96 *Find the volume of the parallelepiped spanned by* $\vec{u} = [1, 2, 3], \vec{v} = [1, 3, 2], \vec{w} = [1, 2, 2]$.

Example 97 *A wrench is used to tighten a bolt. A force of 60 N is applied in a clockwise direction at* $30°$ *to the handle, 20 cm from the centre of the bolt.*

a) Calculate the magnitude of the torque.

b) In what direction does the torque vector point?

Topic 9 Equations of Lines (8.1)

Equations of Lines in $\mathbb{R}^2$: A line in $\mathbb{R}^2$ is determined by a point and a nonzero vector (direction vector) parallel to the line. Let $P(x_0, y_0)$ be a point on the line L. Let $\vec{v} = [v_1, v_2]$ be a nonzero vector which is parallel L.

From Grade 9, you learned: vertical line $x = a$, Slope-intercept form: $y = mx + b$.

- Equation in standard form (also called the scalar equation): $Ax + By + C = 0$.
- Vector equation: $[x, y] = [x_0, y_0] + t[v_1, v_2]$, $t \in \mathbb{R}$.
- Parametric equation: $x = x_0 + tv_1$, $y = y_0 + tv_2$.

Remark. (Two points form). If a line goes through two points P and Q, then $\vec{x}(t) = \vec{p} + t(\vec{q} - \vec{p})$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P.

Example 98 *Find the equation of the line through $P(1, 2)$ and $Q(3, 1)$.*

A **normal vector** to a line L is a vector $\vec{n}$ that is perpendicular to the line.

Example 99 *Find a normal vector to the line through $P(1, 2)$ and $Q(3, 1)$.*

Equations of Lines in $\mathbb{R}^3$: A line in $\mathbb{R}^3$ is determined by a point, $P(x_0, y_0, z_0)$, and a nonzero vector (direction vector) parallel to the line $\vec{v} = [v_1, v_2, v_3]$.

- Vector equation: $[x, y, z] = [x_0, y_0, z_0] + t[v_1, v_2, v_3]$, $t \in \mathbb{R}$.
- Parametric equation: $x = x_0 + tv_1$, $y = y_0 + tv_2$, $z = z_0 + tv_3$.

Example 100 *Find the equation of the line through $P(1, 2, 3)$ and $Q(3, 1, 1)$.*

Topic 10 Equations of Planes (8.2-8.3)

Plane: A plane Π is determined by a point, $P(x_0, y_0, z_0)$, and two non-parallel direction vectors parallel to the plane (or a normal vector $\vec{n}$ which is perpendicular to the plane).

- Vector equation: $\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}$, where
  - $\vec{r} = [x, y, z]$ is a position vector for any point (x, y, z) on the plane,
  - $\vec{r_0} = [x_0, y_0, z_0]$ is the position vector for the given point (x_0, y_0, z_0) on the plane,
  - $\vec{a}$, $\vec{b}$ are two non-parallel direction vectors parallel to the plane,
  - s, t are scalars in $\mathbb{R}$.
- Parametric equations: $x = x_0 + ta_1 + sb_1$, $y = y_0 + ta_2 + sb_2$, $z = z_0 + ta_3 + sb_3$, where $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$.
- Scalar equation (standard equation): $Ax + By + Cz + D = 0$, where $\vec{n} = [A, B, C] = \vec{a} \times \vec{b}$.

Example 101 *Find the equation of the plane through three points $P(1, 2, 3)$, $Q(-3, 2, 1)$, and $R(2, 4, 5)$.*

Example 102 *Determine the scalar equation of each plane.*

a) parallel to the yz-plane; through the point (5, -2, 9).

b) containing the line $[x, y, z] = [1, 2, 4]+t[4, 1, 11]$ and perpendicular to the line $[x, y, z] = [4, 15, 8] + s[2, 3, -1]$.

Topic 11 Intersections of Lines (8.4)

Intersections of Lines in $\mathbb{R}^2$:

Two lines L_1 and L_2 in $\mathbb{R}^2$ can be

- parallel
- intersected

Example 103 *Determine if these lines intersect. If they do, find the coordinates of the point of intersection.* $L_1 : [x, y] = [5, 11] + t[1, 5]$, $L_2 : [x, y] = [3, 2] + s[1, -2]$.

Intersections of Lines in $\mathbb{R}^3$:

Two lines L_1 and L_2 $\mathbb{R}^3$ can be

- parallel
- intersected
- skewed

Example 104 *Determine if these lines intersect. If they do, find the coordinates of the point of intersection.* $L_1 : [x, y, z] = [5, 11, 1] + t[1, 5, 2]$, $L_2 : [x, y, z] = [3, 1, 2] + s[-2, -10, -4]$.

Example 105 *Determine if these lines intersect. If they do, find the coordinates of the point of intersection.* $L_1 : x = 1-t, y = -5+3t, z = 4-t$ *and* $L_2 : x = 2-s, y = 4-s, z = -7+3s$.

Example 106 *Determine if these lines intersect. If they do, find the coordinates of the point of intersection.* $L_1 : [x, y, z] = [5, -4, -2] + t[1, 2, 3]$, $L_2 : [x, y, z] = [2, 0, 1] + s[2, -1, -1]$.

The Distance Between Two Skew Lines:

Let $L_1 : \quad \vec{r} = \vec{p} + t\vec{a}$, $L_2 : \quad \vec{r} = \vec{q} + t\vec{b}$. Let $\vec{n} = \vec{a} \times \vec{b}$, a normal vector perpendicular to L_1 and L_2. Then **the distance between L_1 and L_2** is:

$$d = |proj_{\vec{n}}\overrightarrow{pq}|.$$

Example 107 *Find the distance between* $L_1 : [x, y, z] = [5, -4, -2] + t[1, 2, 3]$ *and* $L_2 : [x, y, z] = [2, 0, 1] + s[2, -1, -1]$.

Topic 12 Intersections of Lines and Planes (8.5)

The Intersections between a Line and a Plane :

Let $\Pi:\ Ax+By+Cz+D=0$, where $\vec{n}=[A,B,C]$ is a normal vector to the plane. Let $L:\ [x,y,z]=[x_0,y_0,z_0]+t\vec{v},\ t\in\mathbb{R}$.

- If $\vec{n}\cdot\vec{v}=0$, then $\Pi//L$. Thus either the line is in the plane, or they have no intersection.
- If $\vec{n}\cdot\vec{v}\neq 0$, then they have only one intersection.

Example 108 *Determine if the line and the plane intersect. If so, determine the solution.*

$$\Pi:\ 9x+13y-2z=29,\qquad L:\ [x,y,z]=[5,-5,2]+t[2,-5,3].$$

Example 109 *Determine if the line and the plane intersect. If so, determine the solution.*

$$\Pi:\ x+3y-4z=10,\qquad L:\ [x,y,z]=[4,-7,1]+t[6,2,3].$$

Example 110 *Determine if the line and the plane intersect. If so, determine the solution.*

$$\Pi:\ 4x-y+11z=-1,\qquad L:\ [x,y,z]=[-2,4,1]+t[3,1,-1].$$

Distance between a point and a plane:

Distance between a point and a plane: A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1,p_2,p_3)$ be a point, let Π be: $ax+by+cz=d$. Then the distance between them is:

$$D=\frac{|ap_1+bp_2+cp_3-d|}{\sqrt{a^2+b^2+c^2}}.$$

Example 111 *Show that the distance between* $P(1,2,3)$ *and the plane* $2x-2y-z=1$ *is 2.*

Topic 13 Intersections of Planes (8.6)

Intersections of two Planes:

There are three possibilities for the intersection of two planes.

- The planes intersect in a line. There are an infinite number of solutions.
- The planes are coincident. There are an infinite number of solutions.
- The planes are parallel and distinct. There is no solution.

Example 112 *Determine if the two planes intersect. If so, determine the solution.*

$$\Pi_1 : \; 2x - y + z = 1, \qquad \Pi_2 : \; x + y + z = 6.$$

Intersections of three Planes:

Three vectors are coplanar if they lie on the same plane. $\vec{n_1}, \vec{n_2}, \vec{n_3}$ are coplanar iff $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) = 0$.

There are three possibilities for the intersection of three planes.

- The planes intersect in a line. $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) \neq 0$, the normals are coplanar and the planes intersect either in a line or not at all.
- The planes intersect in a point. $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) \neq 0$, the normals are not coplanar and the planes intersect in a single point.
- There is no solution. $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) \neq 0$, the normals are coplanar and the planes intersect either in a line or not at all.

Definition 15 *A system of three planes is* **consistent** *if it has one or more solutions. Otherwise, it is inconsistent.*

Example 113 *Determine if the three planes intersect. If so, determine the solution.*

1. $\Pi_1 : \; x + y + z - 1 = 0, \quad \Pi_2 : \; x + 2y + z + 3 = 0, \quad \Pi_3 : \; x + 3y + z + 1 = 0.$
2. $\Pi_1 : \; x + y + z - 1 = 0, \quad \Pi_2 : \; x + 2y + 2z + 3 = 0, \quad \Pi_3 : \; x + 3y + z + 1 = 0.$
3. $\Pi_1 : \; x + y + z + 5 = 0, \quad \Pi_2 : \; x + 2y + z + 3 = 0, \quad \Pi_3 : \; x + 3y + z + 1 = 0.$