Carleton University, Economics 2400C

Assignment 2 Solutions

Due October 3rd, 2016 at the beginning of the class tutorial. The assignment consists of 7 questions (which are all to be completed). Provide a concise but clear explanation of your answers. A reasonable attempt at a solution results in a passing grade. You are encouraged to work together, but you **must submit your own work**.

1. Using the formal "ϵ, δ" definition of limits, show that

$$\lim_{x\to 3}(3x-2)=7$$

Solution Let ϵ be given. To show that the limit of $f(x)$ as $x \to 3$ is 7, we require a value for δ such that $|f(x)-7| < \epsilon$ when $0 < |x-3| < \delta$. Rewrite, $|f(x)-7| = |3x-2-7| = 3|x-3| < \epsilon$. Thus, any value of $\delta \leq \epsilon/3$ will suffice.

2. Compute the following limits

   (a) $\lim_{x\to -2}(x^2+5x)$

   (b) $\lim_{x\to 4} \frac{2x^{3/2}-\sqrt{x}}{x^2-15}$

   (c) $\lim_{x\to a} Ax^n$

Solution

Using $\lim_{x\to a} x = a$ and the rules for limits,

(a) $\lim_{x\to -2}(x^2+5x) = \lim_{x\to -2}(x\cdot x)+\lim_{x\to -2}(5\cdot x) = (\lim_{x\to -2} x)(\lim_{x\to -2} x)+(\lim_{x\to -2} 5)(\lim_{x\to -2} x)$ = (-2)(-2)+5(-2)=-6

(b) $\lim_{x\to 4} \frac{2x^{3/2}-\sqrt{x}}{x^2-15} = \frac{2\lim_{x\to 4} x^{3/2}-\lim_{x\to 4}\sqrt{x}}{\lim_{x\to 4} x^2-15} = \frac{2\cdot 4^{3/2}-\sqrt{4}}{4^2-15} = 14$

(c) $\lim_{x\to a} Ax^n = (\lim_{x\to a} A)(\lim_{x\to a} x^n) = A\cdot(\lim_{x\to a} x)^n = A\cdot a^n$

3. For each of the following functions, indicate at which point(s) the function is discontinuous and explain. In each case graph the function (the domain is $\mathbb{R}$ for all).

   (a)

$$f(x) = \begin{cases} 2x+3, & \text{if } x < 1. \\ x+5, & x \geq 1. \end{cases}$$

   (b)

$$f(x) = 1/x$$

(c)

$$f(x) = 1/(x-3)^2$$

(d)

$$f(x) = (x-2)/(x^2 - x - 2)$$

Solution

(a) The function is not continuous at $x = 1$ since the right hand limit is not equal to the left, i.e. $\lim_{x \to 1^-} f(x) = 5 \neq 6 = \lim_{x \to 1^+} f(x)$.

(b) The function is not conitnuous at $x = 0$ because it is not defined there. Moreover the right hand limit is not equal to the left, i.e. $\lim_{x \to 0^-} f(x) = -\infty \neq +\infty = \lim_{x \to 0^+} f(x)$.

(c) The function is not continuous at $x = 3$ because it is not defined there. However, the right and left side limits are equal (both at $+\infty$).

(d) The function is not continuous at $x = 2$ because it is not defined there. However, the right and left side limits are equal (both at $1/3$). At $x = 1$, neither is the function defined, nor are the right and left limits the same $\lim_{x \to 1^-} f(x) = -\infty \neq +\infty = \lim_{x \to 1^+} f(x)$.

4. Given the function $z = Ax_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, where $x_1, x_2, \dots, x_n > 0$, and $A, a_1, a_2, \dots, a_n$ are all constants with A positive. Compute dz (hint: take the natural logarithm of each side first).

Solution

Taking logs yields,

$$lnz = lnA + a_1 lnx_1 + a_2 lnx_2 + \cdots + a_n lnx_n.$$

Hence,

$$\frac{1}{z} dz = a_1 \frac{1}{x_1} dx_1 + a_2 \frac{1}{x_2} dx_2 + \cdots + a_n \frac{1}{x_n} dx_n$$

so that,

$$dz = z \left(\frac{a_1}{x_1} dx_1 + \frac{a_2}{x_2} dx_2 + \cdots + \frac{a_n}{x_n} dx_n \right).$$

5. Find an expression for dz in terms of dx and dy for the following:

   (a) $z = Ax^a + By^b$

   (b) $z = e^{xu}$, where $u = u(x,y)$.

   (c) $z = ln(x^2 + y)$

Solution

(a) $dz = Ad(x^a) + Bd(y^b) = Aax^{a-1}dx + Bby^{b-1}dy.$

(b) $dz = e^{xu}d(xu) = e^{xu}(xdu + udx) = e^{xu}[x(u_1(x,y)dx + u_2(x,y)dy) + udx]$
$= e^{xu}\{[xu_1(x,y) + u]dx + xu_2(x,y)dy\}$

(c) $dz = dln(x^2 + y) = \frac{d(x^2+y)}{(x^2+y)} = \frac{2xdx+dy}{(x^2+y)}$

6. Consider a utility function $u(x_1, x_2) = x_1^{1/3}x_2^{2/3}$.

   (a) Find the differential du.

   (b) An indifference curve is what is referred to as a level curve of the utility function, which characterizes the relationship between x_1 and x_2 for a fixed level of utility. In other words $u(x_1, x_2) = k$, for a constant k, defines an implicit function $x_2(x_1)$. Characterize the slope of the set of indifference curves using your answer from part (a), by setting $du = 0$ (think about this!). Sketch a few indifference curves.

   (c) Use the implicit function theorem to characterize the slope of indifference curves in part (b), what is the difference between these approaches?

   (d) Repeat parts (a) and (b) for $\tilde{u} = x_1^2x_2^4$. What does this tell you about the two utility functions?

Solution

(a) $du = \frac{1}{3}x_1^{-2/3}x_2^{2/3}dx_1 + \frac{2}{3}x_1^{1/3}x_2^{-1/3}dx_2.$

(b) Setting $du = 0$, characterize the set of indifference curves by simplifying and solving for $\frac{dx_2}{dx_1} = \frac{-x_2}{2x_1}$.

(c) When utility is fixed at constant k, $u(x_1, x_2) - k = 0$ defines an implicit function $x_2(x_1)$ such that $\frac{dx_2}{dx_1}|_{du=0} = -\frac{u_1}{u_2} = \frac{-x_2}{2x_1}$. Which is the same as above when the changes in x_1 and x_2 are infinitesmal. Sketching the curves you should get the standard indifference curve shapes.

(d) The results are the same in that both utility functions have the sames slopes. This is true because $\tilde{u}$ is a monotonic transformation of u. In other words, both functions give the same ordinal ranking of any set of bundles, just assign them different utility numbers. Note that for a given budget constraint, both utility functions give the same optimal consumption bundle (convince yourself of this). As you will learn, it is this ranking of bundles and not the utility number that is relevant for economic theory.

7. The demand for coffee is given by:

$$D_c = 100 - 2p_c + 0.5p_t$$

and for tea:

$$D_t = 120 - p_t + 0.75p_c,$$

where p_c and p_t are the prices of coffee and tea respectively. The respective supply functions are:

$$\begin{aligned} S_c &= 10 + p_c + 5w_c \\ S_t &= 5 + 2p_t + 2w_t, \end{aligned}$$

where w_c and w_t are the indexes of weather conditions affecting production of coffee and tea respectively.

(a) Interpret the supply and demand functions.

(b) Compute the comparative-static effects on equilibrium prices of changes in **both** of the weather condition variables.

Solution

(a) Taking partial derivatives of the demand functions yields

$$\begin{aligned} \frac{\partial D_c}{\partial p_c} &= -2 < 0, \quad & \frac{\partial D_c}{\partial p_t} &= 0.5 > 0 \\ \frac{\partial D_t}{\partial p_t} &= -1 < 0, \quad & \frac{\partial D_t}{\partial p_c} &= 0.75 > 0. \end{aligned}$$

Thus for both tea and coffee, demand is decreasing in own price and increasing in the price of the other. Both goods are substitutes. For supply,

$$\begin{aligned} \frac{\partial S_c}{\partial p_c} &= 1 > 0, \quad & \frac{\partial S_c}{\partial w_c} &= 5 > 0 \\ \frac{\partial S_t}{\partial p_t} &= 2 > 0, \quad & \frac{\partial S_t}{\partial w_t} &= 2 > 0. \end{aligned}$$

(b) The equilibrium condition in the coffee market is $D_c = S_c \Rightarrow 100 - 2p_c^* + 0.5p_t^* = 10 + p_c^* + 5w_c$. In the market for tea $D_t = S_t \Rightarrow 120 - p_t^* + 0.75p_c^* = 5 + 2p_t^* + 2w_t$. This is two equations that characterize our two endogenous variables p_c^* and p_t^*:

$$\begin{aligned} 90 - 3p_c^* + 0.5p_t^* - 5w_c &= 0 \\ 115 - 3p_t^* + 0.75p_c^* - 2w_t &= 0 \end{aligned}$$

Applying the general results for the case of a change in w_c gives the linear system

$$\begin{bmatrix} -3 & 0.5 \\ 0.75 & -3 \end{bmatrix} \begin{bmatrix} \frac{\partial p_c^*}{\partial w_c} \\ \frac{\partial p_t^*}{\partial w_c} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Applying the general results for the case of a change in w_t gives the linear system

$$\begin{bmatrix} -3 & 0.5 \\ 0.75 & -3 \end{bmatrix} \begin{bmatrix} \frac{\partial p_c^*}{\partial w_t} \\ \frac{\partial p_t^*}{\partial w_t} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Solving (using any method you wish) should give

$$\begin{aligned} \frac{\partial p_c^*}{\partial w_c} &\approx -1.739 < 0 \\ \frac{\partial p_t^*}{\partial w_c} &\approx -0.434 < 0 \\ \frac{\partial p_c^*}{\partial w_t} &\approx -0.115 < 0 \\ \frac{\partial p_t^*}{\partial w_t} &\approx -0.6956 < 0. \end{aligned}$$

Better weather conditions for one good results in lower prices for the good itself and the other good, although the amount of the reduction is bigger for the good's own price. Think about why this is here and what assumptions make this so.

1 Extra Practice: Not to be handed in

- Fill out the following table for some calculus practice.

Calculus Practice		
$Y = a + bX$	$\frac{dY}{dX}=$	$\frac{dY}{db}=$
$lnQ = a + blnP$	$\frac{dlnQ}{dlnP}=$	$\frac{dQ}{dP}=$
$I = 500 + 200GDP$	$\frac{dI}{dGDP}=$	
$lnX = a + bt$	$\frac{dlnX}{dt}=$	% growth rate of $X=$
$Q = AP^{-\alpha}$	$\frac{dQ}{dP}=$	Elasticity of Q w.r.t P=
$f(x, y) = x^2y + xlny$	$\frac{\partial f}{\partial x}=$	$\frac{\partial f}{\partial y}=$
$f(x, y, z) = e^z xlny$	$\frac{\partial f}{\partial x}=$	$\frac{\partial^2 f}{\partial z^2}=$
$f(x, y, z) = x^2yz + 2yxz^2$	$\frac{\partial^2 f}{\partial x \partial y}=$	$\frac{\partial^2 f}{\partial y \partial x}=$

Solution:

Calculus Practice		
$Y = a + bX$	$\frac{dY}{dX} = b$	$\frac{dY}{db} = X$
$lnQ = a + blnP$	$\frac{dlnQ}{dlnP} = b$	$\frac{dQ}{dP} = b\frac{Q}{P}$
$I = 500 + 200GDP$	$\frac{dI}{dGDP} = 200$	
$lnX = a + bt$	$\frac{dlnX}{dt} = b$	% growth rate of $X = 100b$
$Q = AP^{-\alpha}$	$\frac{dQ}{dP} = -\alpha AP^{-\alpha-1}$	Elasticity of Q w.r.t P=$-\alpha$
$f(x, y) = x^2y + xlny$	$\frac{\partial f}{\partial x} = 2xy + lny$	$\frac{\partial f}{\partial y} = x^2 + \frac{x}{y}$
$f(x, y, z) = e^z xlny$	$\frac{\partial f}{\partial x} = e^z lny$	$\frac{\partial^2 f}{\partial z^2} = e^z xlny$
$f(x, y, z) = x^2yz + 2yxz^2$	$\frac{\partial^2 f}{\partial x \partial y} = 2xz + 2z^2$	$\frac{\partial^2 f}{\partial y \partial x} = 2xz + 2z^2$

- Given two functions $f(x, y)$ and $g(x, y)$, prove that the product rule holds for differentials (you can assume it holds for derivatives); namely that

$$d[f(x, y)g(x, y)] = f(x, y)dg + g(x, y)df.$$

Solution

$$\begin{aligned} d[f(x, y)g(x, y)] &= \frac{\partial}{\partial x}[f(x, y)g(x, y)]dx + \frac{\partial}{\partial y}[f(x, y)g(x, y)]dy \\ &= [f_x(x, y)g(x, y) + f(x, y)g_x(x, y)]dx + [f_y(x, y)g(x, y) + f(x, y)g_y(x, y)]dy \\ &= f(x, y)[g_x(x, y)dx + g_y(x, y)dy] + g(x, y)[f_x(x, y)dx + f_y(x, y)dy] \\ &= f(x, y)dg + g(x, y)df. \end{aligned}$$

- As in question 6, use the total differential to find the slopes of the indifference curves for each of the following utility functions:

  1. $u(x_1, x_2) = x_1x_2$.

2. $\tilde{u}(x_1, x_2) = x_1^2 x_2^2$.
3. $\hat{u}(x_1, x_2) = B x_1^K x_2^K$, where $B, K > 0$ are constants.

Solution This is a follow up to part (d) from question 6. All three functions are simply monotonic transformations of each other and thus the indifferences curves have the slope $-x_2/x_1$. Of course, part of the exercise is to practice computing differentials so **you** must ensure that this is actually true!