MAT2122 Multivariable Calculus (Fall 2017)

Assignment 1 solutions (77 points)

1 (7 points). Find the equation of the line passing through the points $P = (-5, 0, 4)$ and $Q = (6, -3, 2)$. What is the distance between P and Q?

Solution: By the general formula this equation is

$$\mathbf{v}(t) = \overrightarrow{OP} + t \cdot \overrightarrow{PQ} ,$$ ✓

where

$$\overrightarrow{OP} = (5, 0, 4) ,$$

and

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (6, 3, 2) - (-5, 0, 4) = (11, -3, -2) ,$$ ✓

whence

$$\mathbf{v}(t) = (5, 0, 4) + t \cdot (11, -3, -2) ,$$ ✓

or, in the coordinate form,

$$\begin{cases} x(t) &= -5 + 11t , \\ y(t) &= -3t , \\ z(t) &= 4 - 2t , \end{cases}$$ ✓

where $\mathbf{v}(t) = (x(t), y(t), z(t))$.

The distance between P and Q is the length of the vector $\overrightarrow{PQ}$, ✓ i.e.,

$$\sqrt{11^2 + (-3)^2 + (-2)^2} = \sqrt{134} .$$ ✓

Absence of minor mistakes. ✓

2 (7 points). A triangle has vertices $A = (0, 0, 0), B = (1, 1, 1)$, and $C = (0, 2, 3)$. Find its area.

Solution: The sides of the triangle issued from the vertex $(0, 0, 0)$ are the vectors $(1, 1, 1)$ and $(0, -2, -3)$. ✓ Therefore, the area A of this triangle is equal to one half of the area of the parallelogram spanned by $(1, 1, 1)$ and $(0, -2, -3)$, ✓ i.e.,

$$A = \frac{1}{2}\left\|(1, 1, 1) \times (0, -2, -3)\right\| .$$ ✓

The vector product in question is

$$(1, 1, 1)\times(0, -2, -3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & -2 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = (-1, 3, -2) ,$$ ✓

so that its length is

$$\sqrt{(-1)^2 + (3)^2 + (-2)^2} = \sqrt{1 + 9 + 4} = \sqrt{14} ,$$ ✓

whence

$$A = \frac{\sqrt{14}}{2} .$$ ✓

Absence of minor mistakes. ✓

3 (5 points). Find the volume of the parallelepiped with sides $\mathbf{i}, 3\mathbf{j}-\mathbf{k}$, and $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

Solution: The volume of the parallelepiped spanned by 3 vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the absolute value of the triple product $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ (which does not depend on the order of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$), and coincides with the abolute value of the determinant of the matrix whose rows are the coordinates of the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. In our case $\mathbf{u}=\mathbf{i}, \mathbf{v}=3\mathbf{j}-\mathbf{k}$, and $\mathbf{w}=4\mathbf{i}+2\mathbf{j}-\mathbf{k}$, so that the above matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 4 & 2 & -1 \end{pmatrix} ,$$

and its determinant is $3(-1)-(-1)2=-3+2=-1$, whence the volume is $|-1|=1$.

Absence of minor mistakes.

4 (10 points).

Find an equation for the plane that passes through the points

(a) (0, 0, 0), (2, 0, -1), and (0, 4, -3);
(b) (1, 2, 0), (0, 1, -2), and (4, 0, 1);
(c) (2, -1, 3), (0, 0, 5), and (5, 7, -1).

Solution: **(a)** Let $A=(0,0,0), B=(2,0,-1), C=(0,4,-3)$. Then the plane (ABC) contains the vectors

$$\mathbf{u}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(2,0,-1)-(0,0,0)=(2,0,-1)$$

and

$$\mathbf{v}=\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(0,4,-3)-(0,0,0)=(0,4,-3) ,$$

so that it is orthogonal to the vector

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix}=\mathbf{i}\begin{vmatrix} 0 & -1 \\ 4 & -3 \end{vmatrix}-\mathbf{j}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix}+\mathbf{k}\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}=4\mathbf{i}+6\mathbf{j}+8\mathbf{k}=(4,6,8) .$$

Thus, a point $P=(x,y,z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(x,y,z)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}'=(2,3,4)$, i.e., if and only if $\overrightarrow{AP}\cdot\mathbf{w}'=0$. Therefore,

$$2x+3y+4z=0$$

is an equation of the plane (ABC).

(b) For $A=(1,2,0), B=(0,1,-2)$, and $C=(4,0,1)$ the plane contains the vectors

$$\mathbf{u}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(0,1,-2)-(1,2,0)=(-1,-1,-2)$$

and

$$\mathbf{v}=\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4,0,1)-(1,2,0)=(3,-2,1) ,$$

so that the plane is orthogonal to the vector

$$\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix}=\mathbf{i}\begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix}-\mathbf{j}\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}+\mathbf{k}\begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix}=-5\mathbf{i}-5\mathbf{j}+5\mathbf{k}=(-5,-5,5) .$$

Thus, a point $P=(x,y,z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(x-1,y-2,z)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}' = (1, 1, -1)$, i.e., if and only if $\overrightarrow{AP} \cdot \mathbf{w}' = 0$. Therefore,

$$(x-1) + (y-2) - 5z = x + y - z - 3 = 0$$

is an equation of the plane (ABC).

(c) For $A = (2, -1, 3)$, $B = (0, 0, 5)$, and $C = (5, 7, -1)$ the plane contains the vectors

$$\mathbf{u} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0, 0, 5) - (2, -1, 3) = (-2, 1, 2)$$

and

$$\mathbf{v} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (5, 7, -1) - (2, -1, 3) = (3, 8, -4) ,$$

so that the plane is orthogonal to the vector

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 3 & 8 & -4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 2 \\ 8 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 2 \\ 3 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 1 \\ 3 & 8 \end{vmatrix} = -20\mathbf{i} - 2\mathbf{j} - 19\mathbf{k} = (-20, -2, -19) .$$

Thus, a point $P = (x, y, z)$ belongs to the plane (ABC) if and only if the vector

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x - 2, y + 1, z - 3)$$

is orthogonal to $\mathbf{w}$, or, equivalently, to $\mathbf{w}' = (20, 2, 19)$, i.e., if and only if $\overrightarrow{AP} \cdot \mathbf{w}' = 0$. Therefore,

$$20(x - 2) + 2(y + 1) + 19(z - 3) = 20x + 2y + 19z - 95 = 0$$

is an equation of the plane (ABC).

Absence of minor mistakes.

5 (7 points).

Find the distance from the point $P = (2, 2, 0)$ to the line passing through the points $A = (0, 0, -1)$ and $B = (-1, 1, 0)$.

Solution: The directing vector of the line is

$$\mathbf{v} = \overrightarrow{OB} - \overrightarrow{OA} = (-1, 1, 0) - (0, 0, -1) = (-1, 1, 1) .$$

Since A belongs to the line, the distance in question is the length of the vector

$$\overrightarrow{AP} - pr_{\mathbf{v}} \overrightarrow{AP} ,$$

where

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (2, 2, 0) - (0, 0, -1) = (2, 2, 1) .$$

Since

$$pr_{\mathbf{v}} \overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{(2, 2, 1) \cdot (-1, 1, 1)}{(-1, 1, 1) \cdot (-1, 1, 1)} (-1, 1, 1) = \frac{1}{3}(-1, 1, 1) = \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) ,$$

whence

$$\overrightarrow{AP} - pr_{\mathbf{v}} \overrightarrow{AP} = (2, 2, 1) - \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{7}{3}, \frac{5}{3}, \frac{2}{3}\right) ,$$

and the distance in question is the length of the vector $\left(\frac{7}{3}, \frac{5}{3}, \frac{2}{3}\right)$, i.e. $\frac{\sqrt{78}}{3}$.

Absence of minor mistakes.

6 (6 points).

Find the distance from the point $P = (1,4,0)$ to the plane passing through the points $A = (0,0,0), B = (2,0,-1), C = (2,-1,0)$.

Solution: The vector product of the vectors $\overrightarrow{AB} = (2,0,-1)$ and $\overrightarrow{AC} = (2,-1,0)$ is

$$\mathbf{n} = (2,0,-1)\times(2,-1,0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} = -\mathbf{i}-2\mathbf{j}-2\mathbf{k} = (-1,-2,-2) ,$$

which is normal to the plane (ABC). Therefore the distance in question coincides with the length of the projection of the vector $\overrightarrow{AP} = (1,4,0) - (0,0,0) = (1,4,0)$ to the direction of the vector $\mathbf{n}$. Since

$$pr_{\mathbf{n}}\overrightarrow{AP} = \frac{\mathbf{n}\cdot\overrightarrow{AP}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{(-1,-2,-2)\cdot(1,4,0)}{(-1,-2,-2)\cdot(-1,-2,-2)}(-1,-2,-2) = (1,2,2) ,$$

the distance in question is $\sqrt{9} = 3$.

Absence of minor mistakes.

7 (4 points). Find the partial derivatives of the function $z(x,y) = \sqrt{a^2 - x^2 - y^2}$ at the points $(0,0)$ and $(a/2, a/2)$.

Solution:

$$\frac{\partial z}{\partial x}(x,y) = \frac{\partial\sqrt{a^2-x^2-y^2}}{\partial x} = -\frac{2x}{2\sqrt{a^2-x^2-y^2}} = -\frac{x}{\sqrt{a^2-x^2-y^2}} .$$

In precisely the same way

$$\frac{\partial z}{\partial y}(x,y) = -\frac{y}{\sqrt{a^2-x^2-y^2}} ,$$

whence

$$\frac{\partial z}{\partial x}(0,0) = \frac{\partial z}{\partial y}(0,0) = 0 ,$$

$$\frac{\partial z}{\partial x}(a/2,a/2) = \frac{\partial z}{\partial y}(a/2,a/2) = -\frac{a/2}{\sqrt{a^2-(a/2)^2-(a/2)^2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} .$$

Absence of minor mistakes.

8 (4 points). Find the equation of the plane tangent to the graph of the function $f(x,y) = x\cos x\cos y$ at the point $(0,\pi)$.

Solution: The partial derivatives of the function f are

$$f'_x(x,y) = (x\cos x\cos y)'_x = \cos y(x\cos x)'_x = \cos y(\cos x - x\sin x)$$

and

$$f'_y(x,y) = (x\cos x\cos y)'_y = -x\cos x\sin y ,$$

so that for the point $(x_0,y_0) = (0,\pi)$

$$f(x_0,y_0) = 0 , \qquad f'_x(x_0,y_0) = -1 , \qquad f'_y(x_0,y_0) = 0 .$$

Therefore, the tangent plane equation is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0) = -x \ .$$

Absence of minor mistakes.

9 (6 points). A particle following the path $\mathbf{c}(t) = (e^t, e^{-t}, \cos t)$ flies off on a tangent at $t = t_0 = 1$. Compute the position of the particle at time $t_1 = 2$.

Solution: The equation of the considered tangent line $\mathbf{l}$ is

$$\mathbf{l}(t) - \mathbf{c}(t_0) = (t - t_0)\dot{\mathbf{c}}(t_0) \ ,$$

or

$$\mathbf{l}(t) = \mathbf{c}(t_0) + (t - t_0)\dot{\mathbf{c}}(t_0) \ .$$

In our case $t_0 = 1$, and

$$\mathbf{c}(t_0) = \mathbf{c}(1) = (e, e^{-1}, \cos 1) \ .$$

On the other hand,

$$\dot{\mathbf{c}}(t) = (e^t, -e^{-t}, -\sin t) \ ,$$

so that

$$\dot{\mathbf{c}}(t_0) = \dot{\mathbf{c}}(1) = (e, -e^{-1}, -\sin 1) \ ,$$

whence

$$\mathbf{l}(2) = (e, e^{-1}, \cos 1) + (e, -e^{-1}, -\sin 1) = (2e, 0, \cos 1 - \sin 1) \ .$$

Absence of minor mistakes.

10 (5 points). Compute the matrix of partial derivatives of the functions

(a) $f(x, y) = (e^x, \sin xy)$,
(b) $f(x, y, z) = (x - y, y + z)$,
(c) $f(x, y) = (x + y, x - y, xy)$,
(d) $f(x, y, z) = (x + z, y - 5z, x - y)$.

Solution: **(a)**

$$\mathbf{D}f(x, y) = \begin{pmatrix} \dfrac{\partial e^x}{\partial x} & \dfrac{\partial e^x}{\partial y} \\ \dfrac{\partial(\sin xy)}{\partial x} & \dfrac{\partial(\sin xy)}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ y\cos xy & x\cos xy \end{pmatrix}$$

(b)

$$\mathbf{D}f(x, y, z) = \begin{pmatrix} \dfrac{\partial(x-y)}{\partial x} & \dfrac{\partial(x-y)}{\partial y} & \dfrac{\partial(x-y)}{\partial z} \\ \dfrac{\partial(y+z)}{\partial x} & \dfrac{\partial(y+z)}{\partial y} & \dfrac{\partial(y+z)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(c)

$$\mathbf{D}f(x,y) = \begin{pmatrix} \frac{\partial(x+y)}{\partial x} & \frac{\partial(x+y)}{\partial y} \\ \frac{\partial(x-y)}{\partial x} & \frac{\partial(x-y)}{\partial y} \\ \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{pmatrix}$$ ✓

(d)

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \frac{\partial(x+z)}{\partial x} & \frac{\partial(x+z)}{\partial y} & \frac{\partial(x+z)}{\partial z} \\ \frac{\partial(y-5z)}{\partial x} & \frac{\partial(y-5z)}{\partial y} & \frac{\partial(y-5z)}{\partial z} \\ \frac{\partial(x-y)}{\partial x} & \frac{\partial(x-y)}{\partial y} & \frac{\partial(x-y)}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & -1 & 0 \end{pmatrix}$$ ✓

Absence of minor mistakes. ✓

11 (9 points).

One tracks the values of the function $f(x,y,z) = xy - 2yz^2$ along the curve $\mathbf{c}(t) = (t^2, t, -t)$. How fast does it change at $t = 1$? What is the direction of the fastest decrease of the function f at the point $\mathbf{c}(1)$?

Solution: By the chain rule the derivative of the composite function $f \circ \mathbf{c}(t)$ at a point t is

$$(f \circ \mathbf{c})'(t) = \nabla f(\mathbf{c}(t)) \cdot \dot{\mathbf{c}}(t) \ ,$$ ✓

where

$$\dot{\mathbf{c}}(t) = (2t, 1, -1)$$ ✓

and

$$\nabla f(x,y,z) = (y, x - 2z^2, -4yz) \ .$$ ✓

At the point $t = 1$

$$\dot{\mathbf{c}}(1) = (2, 1, -1) \ ,$$ ✓

and, since $\mathbf{c}(1) = (1, 1, -1)$, ✓

$$\nabla f(\mathbf{c}(1)) = \nabla f((1,1,-1)) = (1, -1, 4) \ ,$$ ✓

whence

$$(f \circ \mathbf{c})'(1) = \nabla f(\mathbf{c}(1)) \cdot \dot{\mathbf{c}}(1) = (1,-1,4) \cdot (2,1,-1) = -3 \ .$$ ✓

The direction of the fastest decrease of f at the point $\mathbf{c}(1) = (1,1,-1)$ is

$$-\nabla f(1,1,-1) = (-1, 1, -4) \ .$$ ✓

Absence of minor mistakes. ✓

12 (7 points). Let $g(u,v) = (e^u, u + sinv)$ and $f(x,y,z) = (xy, yz)$. Compute $\mathbf{D}(g \circ f)$ at $(0,1,0)$ using the chain rule.

Solution: The derivatives of the functions g and f are, respectively

$$\mathbf{D}g(u,v) = \begin{pmatrix} \frac{\partial e^u}{\partial u} & \frac{\partial e^u}{\partial v} \\ \frac{\partial (u+\sin v)}{\partial u} & \frac{\partial (u+\sin v)}{\partial v} \end{pmatrix} = \begin{pmatrix} e^u & 0 \\ 1 & \cos v \end{pmatrix}$$

and

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \frac{\partial xy}{\partial x} & \frac{\partial xy}{\partial y} & \frac{\partial xy}{\partial z} \\ \frac{\partial yz}{\partial x} & \frac{\partial yz}{\partial y} & \frac{\partial yz}{\partial z} \end{pmatrix} = \begin{pmatrix} y & x & 0 \\ 0 & z & y \end{pmatrix}.$$

The evaluation of f at the point $(0,1,0)$ and of g at the point

$$f(0,1,0) = (0,0)$$

gives

$$\mathbf{D}f(0,1,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\mathbf{D}g(0,0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

whence

$$\mathbf{D}(g \circ f)(0,1,0) = \mathbf{D}g(0,0)\mathbf{D}f(0,1,0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Absence of minor mistakes.