Concordia University

EMAT 213 - Final Exam

Instructor: Dr. Bertola

Course Examiner: Dr. R. Stern

Date: June 2003.

Time allowed: 3 hours.

Directions: NO CALCULATORS.

This exam has two pages!

[40 pts] Problem 1

Solve the following *first* order ODEs by finding the general solution and the solution of the IVP (when given).

$$\textbf{(a)} \qquad (x^2+3x+8)\,y' = (y-1)(2x+3)\ , \qquad y(0)=2$$

$$\textbf{(b)} \qquad \left(y^2+yx\right)\mathrm{d}x + x^2\mathrm{d}y = 0\ ; \qquad y(1)=2$$

$$\textbf{(c)} \qquad x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy = y^4$$

$$\textbf{(d)} \qquad \left(y^2\cos(x) - 3x^2y - 2x\right)\mathrm{d}x + \left(2y\sin(x) - x^3 + \ln(y)\right)\mathrm{d}y$$

[20 pts] Problem 2

Solve the following linear ODEs by finding the general solution

$$\textbf{(a)} \qquad y^{(4)} + y'' - 12y = x^3\ ;$$

$$\textbf{(b)} \qquad x^2y'' - 12y = \ln(x)\ , \qquad (x>0).$$

[6 pts] Problem 3

Show that the following power series

$$y = \sum_{n=1}^{\infty}\frac{1}{n\,2^n}x^n$$

is a solution of the ODE

$$(2-x)y'' - y' = 0\ .$$

[10 pts] Problem 4

(i)[9pt] Find the general solution as a power series centered at the regular point $x=0$ and up to degree 5 of the differential equation below.

(ii)[1pt] What is the **minimum radius of convergence** of the solution written as a power series?

$$(x^2+4)y'' + y = 0$$

[14 pts] Problem 5

Consider the following *nonhomogeneous linear* system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = & & y & +t \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = & x & & +\mathrm{e}^t \end{cases}$$

(a)[7pt] Rewrite the system in matrix form and find the general solution, X_c of the associated homogeneous system.
(b)[7pt] Find a particular solution X_p of the system, using the technique of variation of parameters.

[10 pts] Problem 6

A spring is stretched by one meter by a force of 5 Newtons.
A mass of one Kilogram is attached to the loose end and it is released from the equilibrium position without any initial velocity. Find the motion of the mass knowing that there is a damping coefficient equal to 2 $N \times s/m$ and that the mass is subject to an external force of equation

$$F(t) = 6$$

[*Note that you have to solve also the IVP implied by the text above*]

[5 pts] Bonus Problem 1

Sketch a qualitative graph for $t \geq 0$ of the motion of the mass in Problem 6. The scale of the axes is unimportant but you must clearly indicate the slope of the tangent at $t = 0$, the initial position, the asymptotic behavior for $t \to \infty$ and the oscillatory behavior if any.

[5 pts] Bonus Problem 2

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = P(x) + Q(x)y + R(x)y^2$$

is known as **Riccati's equation**.
Suppose that one solution y_1 is known; **show** that the substitution

$$y = y_1 + u$$

(where $u = u(x)$ is a new unknown function) yields a *Bernoulli* equation for $u(x)$ with $n = 2$. **You do not have to solve it**.