Solutions Attached.

MATH 204/4 all sections except EC: - VECTORS AND MATRICES
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Alternate Midterm - Saturday, March 8, 2014 (1h30min) 10:00 - 11:30
Only approved calculators are permitted.

Justify all your answers.

1. Solve by the Gauss-Jordan elimination

$$\begin{aligned} x - 3y + 2z - s + 2t &= 2 \\ 3x - 9y + 7z - s + 3t &= 7 \\ 2x - 6y + 7z + 4s - 5t &= 7 \end{aligned}$$

2. Determine the values of k so that the following system has
   (i) a unique solution
   (ii) no solution
   (iii) an infinite number of solutions

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

3. If $(6A - 4I)^{-1} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$, find A.

4. Find the inverse of A if A $= \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}$.

5. Find the determinant of $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{pmatrix}$.

6. Write the vector $V = (2, -5, 3)$ as a linear combination of

   $U_1 = (1, -3, 2)$, $U_2 = (2, -4, -1)$, $U_3 = (1, -5, 7)$.

Math 204
Mid term Alternate March 2014

Question 1

ACM:

$$\begin{matrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{matrix} \qquad R2 = R2 + (-3)*R1$$

$$\begin{matrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{matrix} \qquad R3 = R3 + (-2)*R1$$

$$\begin{matrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 3 & 6 & -9 & 3 \end{matrix} \qquad R1 = R1 + (-2)*R2$$

$$\begin{matrix} 1 & -3 & 0 & -5 & 8 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 3 & 6 & -9 & 3 \end{matrix} \qquad R3 = R3 + (-3)*R2$$

RRE:

$$\begin{matrix} 1 & -3 & 0 & -5 & 8 & 0 & (1) \\ 0 & 0 & 1 & 2 & -3 & 1 & (2) \\ 0 & 0 & 0 & 0 & 0 & 0 & (3) \end{matrix}$$

(3) $\Rightarrow$ Infinite Number solutions

(2) $\Rightarrow z + 2s - 3t = 1 \Rightarrow z = 1 - 2s + 3t$

(1) $\Rightarrow x - 3y - 5s + 8t = 0 \Rightarrow x = 3y + 5s - 8t$

Let variable $y = p_1$, $s = p_2$, $t = p_3$

$\Rightarrow$ $z = 1 - 2p_2 + 3p_3$

$x = 3p_1 + 5p_2 - 8p_3$

Question 2

ACM:

$$\begin{matrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{matrix} \qquad R1 <-> R3$$

$$\begin{matrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{matrix} \qquad R2 = R2 + (-1)*R1$$

$$\begin{matrix} 1 & 1 & k & 1 \\ 0 & -1+k & 1-k & 0 \\ k & 1 & 1 & 1 \end{matrix} \qquad R3 = R3 + (-k)*R1$$

$$\begin{matrix} 1 & 1 & k & 1 \\ 0 & -1+k & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{matrix} \qquad R3 = R3 + (1)*R2$$

$$\begin{matrix} 1 & 1 & k & 1 \\ 0 & -1+k & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k & (3) \end{matrix}$$

In Row (3) $\Rightarrow$ $0 \quad 0 \quad (2+k)(1-k) \quad (1-k)$

Case I If $k = 1$ Row 3 becomes $0 \; 0 \; 0 \; 0$
$\Rightarrow$ Infinite N°. solutions

Case II If $k = -2$ Row 3 becomes $0 \; 0 \; 0 \; 3$
$\Rightarrow$ No solution

Case III If $k \neq 1$ and $k \neq -2$ then
we can do more ERO Starting with
$R3 = \frac{1}{(2+k)(1-k)} R_3$ and we get
unique solution.

Question 3

Let $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = B$: Solve for A:

$$(6A - 4I)^{-1} = B$$

$$((6A-4I)^{-1})^{-1} = B^{-1}$$

$$6A - 4I = B^{-1}$$

$$6A = B^{-1} + 4I$$

$$A = \frac{1}{6}(B^{-1} + 4I)$$

$$A = \frac{1}{6}\left(\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}^{-1} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{(3)(5)-(2)(7)}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{6}\left(\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{6}\begin{bmatrix} 9 & -7 \\ -2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{6} & -\frac{7}{6} \\ -\frac{2}{6} & \frac{7}{6} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{7}{6} \\ -\frac{1}{3} & \frac{7}{6} \end{bmatrix}$$

Question 4

Find $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}^{-1}$:

```
1  -2  2   1   0   0
2  -3  6   0   1   0
1   1  7   0   0   1    R2 = R2 + (-2)*R1

1  -2  2   1   0   0
0   1  2  -2   1   0
1   1  7   0   0   1    R3 = R3 + (-1)*R1

1  -2  2   1   0   0
0   1  2  -2   1   0
0   3  5  -1   0   1    R1 = R1 + (2)*R2

1   0  6  -3   2   0
0   1  2  -2   1   0
0   3  5  -1   0   1    R3 = R3 + (-3)*R2

1   0  6  -3   2   0
0   1  2  -2   1   0
0   0 -1   5  -3   1    R3 = (-1)*R3

1   0  6  -3   2   0
0   1  2  -2   1   0
0   0  1  -5   3  -1    R1 = R1 + (-6)*R3

1   0  0  27 -16   6
0   1  2  -2   1   0
0   0  1  -5   3  -1    R2 = R2 + (-2)*R3

1   0  0  27 -16   6
0   1  0   8  -5   2
0   0  1  -5   3  -1
```

$$\Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Question 5.

$$\det A = \begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} \quad R2 = R2 + (1)*R1$$

$$= \begin{vmatrix} 1 & 2 & 2 & 3 \\ 2 & 2 & 0 & 3 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} \quad R1 = R1 + (-2)*R3$$

$$= \begin{vmatrix} -5 & 4 & 0 & 7 \\ 2 & 2 & 0 & 3 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix}$$

Now cofactor expansion about Column 3:

$$0 + 0 + 1(-1)^{3+3} \begin{vmatrix} -5 & 4 & 7 \\ 2 & 2 & 3 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= \begin{matrix} -5 & 4 & 7 & -5 & 4 \\ 2 & 2 & 3 & 2 & 2 \\ 4 & -3 & 2 & 4 & -3 \end{matrix}$$

$$= (-5)(2)(2) + (4)(3)(4) + (7)(2)(-3) - (4)(2)(7) - (-3)(3)(-5) - (2)(2)(4)$$

$$= -131$$

Question 6: Assume there exists scalars k_1, k_2, k_3 such that:

$$k_1\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + k_3\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} \Rightarrow \begin{aligned} k_1 + 2k_2 + k_3 &= 2 \\ -3k_1 - 4k_2 - 5k_3 &= -5 \\ 2k_1 - k_2 + 7k_3 &= 3 \end{aligned}$$

Solve for k_i using GAUSS-JORDAN

$$\begin{matrix} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{matrix} \quad R2 = R2 + (3)*R1$$

$$\begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 2 & -1 & 7 & 3 \end{matrix} \quad R3 = R3 + (-2)*R1$$

$$\begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{matrix} \quad R1 = R1 + (-1)*R2$$

$$\begin{matrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{matrix} \quad R2 \text{ <-> } R3$$

$$\begin{matrix} 1 & 0 & 3 & 1 \\ 0 & -5 & 5 & -1 \\ 0 & 2 & -2 & 1 \end{matrix} \quad R2 = R2 + (3)*R3$$

$$\begin{matrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 1 \end{matrix} \quad R3 = R3 + (-2)*R2$$

$$\begin{matrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -3 \end{matrix} \quad (3)$$

(3) $\Rightarrow$ No solution exists for k_i

$\Rightarrow$ It is not possible to write V as a linear combination: $k_1 u_1 + k_2 u_2 + k_3 u_3$