STAT 349 Midterm Exam Solutions (Fall 2016)

1. A fair six-sided die is continually tossed. Find the probability that the number 6 appears before any odd number. [**10** marks]

   Solution:

   Method 1: Symmetry

   Define for $i = 1, 3, 5, 6$ the event

   $$A_i := \{\text{The number } i \text{ will be the first among } \{1,3,5,6\} \text{ to appear.}\}$$

   We have by symmetry that

   $$\begin{aligned} \mathbb{P}[A_1] &= \mathbb{P}[A_3] = \mathbb{P}[A_5] = \mathbb{P}[A_6], \\ 1 &= \mathbb{P}[A_1] + \mathbb{P}[A_3] + \mathbb{P}[A_5] + \mathbb{P}[A_6]. \end{aligned}$$

   Thus $\mathbb{P}[A_6] = 1/4$.

   Method 2: Conditioning on the final throw

   Let N be the throw at which either an odd number or 6 occurs. Then,

   $$\begin{aligned} \mathbb{P}[A_6] &= \sum_{n=1}^{\infty} \mathbb{P}[A_6, N = n] \\ &= \sum_{n=1}^{\infty} \mathbb{P}[\{\text{get 6 on throw } n \text{ and either 2 or 4 on throws } 1, \dots, n-1\}] \\ &= \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{6}\right)^{n-1} \quad \text{since throws are independent} \\ &= \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{2}{6}\right)^{n} = \frac{1}{6}\frac{1}{1-\frac{2}{6}} = \frac{1}{6}\frac{6}{6-2} = \frac{1}{4}. \end{aligned}$$

   Method 3: Conditioning on the outcome of the first toss

   We use A to denote the event that the number 6 appears before any odd number. Then

   $$P(A) = P(A|6)\cdot\frac{1}{6} + P(A|2 \text{ or } 4)\cdot\frac{2}{6} + P(A|1,3, \text{ or } 5)\cdot\frac{3}{6} = \frac{1}{6} + P(A)\cdot\frac{2}{6} + 0.$$

   Solving the equation $P(A) = \frac{1}{6} + P(A)\cdot\frac{2}{6}$, we obtain that $P(A) = \frac{1}{4}$.

2. Assume the number of accidents per day at the industrial plant UnsafeShopInc follows a Poisson distribution with parameter $\lambda = 3$. The numbers of accidents which occur during two different days are independent.

Furthermore, every time an accident occurs, the number of workers injured follows a binomial distribution with parameters $n = 4, p = 0.2$. The numbers of workers injured during two different accidents are independent.

Assume there are five working days in a week.

Question (a): Denote by N the total number of injured people at the plant during a given week. Find $\mathbb{E}[N]$ and $\mathrm{Var}(N)$.

Solution:

$$\begin{aligned}
N &= \sum_{i=1}^{5} S_i, \\
S_i &= \sum_{j=1}^{M_i} X_{i,j} \\
M_i &\sim \text{Poisson}(\lambda = 3) \\
X_{i,j} &\sim \text{Binomial}(n = 4, p = 0.2)
\end{aligned}$$

M_i and $X_{i,j}$ are independent for all i, j.

$$\begin{aligned}
\mathbb{E}[S_i] &= \mathbb{E}[M_i]\,\mathbb{E}[X_{i,j}] = \lambda np = 3 \times (4 \times 0.2) = 2.4. \\
Var[S_i] &= Var[X_{i,j}]\,\mathbb{E}[M_i] + \mathbb{E}[X_{i,j}]^2\, Var[M_i] \\
&= np(1-p)\lambda + (np)^2\,\lambda \\
&= 4 \times 0.2 \times 0.8 \times 3 + (4 \times 0.2)^2 \times 3 \\
&= 3.84.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[N] &= \mathbb{E}\left[\sum_{i=1}^{5} S_i\right] = \sum_{i=1}^{5} \mathbb{E}[S_i] = 5\times = 2.4 = 12, \\
Var[N] &= Var\left[\sum_{i=1}^{5} S_i\right] = \sum_{i=1}^{5} Var[S_i] = 5 \times 3.84 = 19.2.
\end{aligned}$$

where the second line holds because S_i and S_j are independent if $i \neq j$.

Question (b): What is the probability that no workers get injured during a given day?

[**6** + **4** marks]

Solution:

$$\begin{aligned}
\mathbb{P}[S_i = 0] &= \mathbb{P}\left[\sum_{j=1}^{M_i} X_{i,j} = 0\right] \\
&= \sum_{m=0}^{\infty} \mathbb{P}[X_{i,1} = \ldots = X_{i,m} = 0 | M_i = m] \, \mathbb{P}[M_i = m] \\
&= \sum_{m=0}^{\infty} \mathbb{P}[X_{i,j} = 0]^m \, e^{-\lambda} \frac{\lambda^m}{m!} \\
&= \sum_{m=0}^{\infty} [(1-p)^4]^m e^{-\lambda} \frac{\lambda^m}{m!} \\
&= \sum_{m=0}^{\infty} e^{-\lambda} \frac{((1-p)^4\lambda)^m}{m!} \\
&= \frac{e^{-\lambda}}{e^{-(1-p)^4\lambda}} \underbrace{\sum_{m=0}^{\infty} e^{-(1-p)^4\lambda} \frac{((1-p)^4\lambda)^m}{m!}}_{=1} \\
&= \frac{e^{-\lambda}}{e^{-(1-p)^4\lambda}} = e^{-\lambda + \lambda(1-p)^4} = 0.17013.
\end{aligned}$$

3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a discrete-time Markov chain with the transition probability matrix for respective states $\mathcal{S} = \{0, 1, 2, 3\}$ given by

$$P = [P_{i,j}]_{i,j\in\mathcal{S}} = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.4 & 0.25 & 0.25 \\ 0.5 & 0.1 & 0.4 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix},$$

where $P_{i,j} := \mathbb{P}[X_{n+1} = j | X_n = i]$. Let $\mathcal{A} := \{2, 3\}$.

[**5** + **5** marks]

Question (a): Compute $\mathbb{P}[X_3 = 1, X_k \notin \mathcal{A} \text{ for all } k \in \{1, 2\} | X_0 = 0]$.

Solution:

Method 1: Direct computation

$$\begin{aligned}
& \mathbb{P}[X_3 = 1, X_k \notin \mathcal{A} \text{ for all } k \in \{1, 2\} | X_0 = 0] \\
= \; & \mathbb{P}[X_3 = 1, X_2 = 0, X_1 = 0 | X_0 = 0] + \mathbb{P}[X_3 = 1, X_2 = 1, X_1 = 0 | X_0 = 0] \\
& + \mathbb{P}[X_3 = 1, X_2 = 0, X_1 = 1 | X_0 = 0] + \mathbb{P}[X_3 = 1, X_2 = 1, X_1 = 1 | X_0 = 0] \\
= \; & P_{0,0}P_{0,0}P_{0,1} + P_{0,0}P_{0,1}P_{1,1} + P_{0,1}P_{1,0}P_{0,1} + P_{0,1}P_{1,1}P_{1,1} = 0.093.
\end{aligned}$$

Method 2: Q matrix

Define a new Markov chain W with states $\tilde{\mathcal{S}} = \{0, 1, A\}$ and transition matrix Q:

$$Q = [Q_{i,j}]_{i,j\in\tilde{\mathcal{S}}} = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

We get

$$Q^2 = \begin{bmatrix} 0.07 & 0.18 & 0.75 \\ 0.06 & 0.19 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 0.032 & 0.093 & 0.875 \\ 0.031 & 0.094 & 0.875 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\mathbb{P}\left[X_3 = 1, X_k \notin \mathcal{A} \text{ for all } k \in \{1, 2\} | X_0 = 0\right] = Q^{(3)}_{0,1} = 0.093$.

Question (b): Let $N := \min\{n \in \mathbb{N} : X_n \in \mathcal{A}\}$. Compute $\mathbb{E}[N|X_0 = 0]$.

Solution:

Method 1:

Define $f_i := \mathbb{E}[N|X_0 = i]$.

Clearly, $f_2 = f_3 = 0$. Furthermore, for $i \notin \mathcal{A}$, conditioning upon the first step,

$$\begin{aligned} f_i &= \sum_{k=0}^{3}(f_k + 1)P_{i,k} = f_0 P_{i,0} + f_1 P_{i,1} + 1. \\ \Rightarrow f_0 &= \frac{f_1(1 - P_{1,1}) - 1}{P_{1,0}} = 6f_1 - 10, \\ \text{and } f_0 &= \frac{f_1 P_{0,1} + 1}{1 - P_{0,0}} = f_1\frac{3}{8} + \frac{10}{8}. \\ \Rightarrow 48f_1 - 80 &= 3f_1 + 10 \\ \Rightarrow f_1 &= \frac{90}{45} = 2 \\ \Rightarrow f_0 &= 6f_1 - 10 = 12 - 10 = 2. \end{aligned}$$

Thus the answer is $\mathbb{E}[N|X_0 = 0] = f_0 = 2$.

Method 2:

From the transition matrix P, we find that if $X_0 = 0$, then $X_1 \in \mathcal{A} = \{2, 3\}$ with probability $0.4 + 0.1 = 0.5$, and $X_1 \in \{0, 1\}$ with probability $0.2 + 0.3 = 0.5$; if $X_0 = 1$, then $X_1 \in \mathcal{A} = \{2, 3\}$ with probability $0.25 + 0.25 = 0.5$, and $X_1 \in \{0, 1\}$ with probability $0.1 + 0.4 = 0.5$. Then, if the Markov chain starts with 0 or 1, the number of time units that it takes the Markov chain to enter $\mathcal{A}$ for the first time has a geometric distribution with probability 0.5. Therefore, the desired conditional expectation is equal to 1/(0.5)=2.

4. Two red balls and two black balls are distributed in two urns, with two balls in each urn. At each step, we first draw a ball from urn 1 at random and put it into urn 2. Then we draw a ball randomly from the **three** balls in urn 2, and put it back to urn 1. So after each step each urn contains exactly two balls. Let X_n be the total number of red balls in urn 1 at the beginning of the n-th step. $\{X_n\}$ is a Markov chain. Obtain the transition probability matrix of this Markov chain and find the proportion of time the chain spends in each state in the long run. [**10** marks]

   Solution:

   The transition probability matrix (for states 0, 1 and 2) is

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

   The stationary distribution is obtained through $\pi_0 + \pi_1 + \pi_2 = 1$ and $\pi = \pi P$, which is

$$\begin{aligned} \pi_0 &= 1/3\pi_0 + 1/6\pi_1, \\ \pi_1 &= 2/3\pi_0 + 2/3\pi_1 + 2/3\pi_2, \\ \pi_2 &= 1/6\pi_1 + 1/3\pi_2, \end{aligned}$$

   which gives

$$\pi_0 = \pi_2 = 1/6, \quad \pi_1 = 4/6.$$

   π_i is the long-term proportion of time the chain spends in state i for all i.