MAT8201 – Test 1

Name: Solutions

$\frac{\quad}{40}$

1) Find the following limits: **(2 marks each)**

a) $\lim_{x \to 5} \frac{x^2-5x}{x-5}$

$= \lim_{x \to 5} \frac{x\cancel{(x-5)}}{\cancel{x-5}}$

$= \lim_{x \to 5} x = \boxed{5}$

b) $\lim_{x \to 2} \frac{x^2-x-6}{x-3}$

$= \frac{2^2-2-6}{2-3} = \frac{-4}{-1}$

$= \boxed{4}$

2) Find the derivative from first principles (in other words use limits): **(3 marks each)**

a) $y = x^2 - 2$

$y' = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2-2-(x^2-2)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= \lim_{h \to 0} 2x+h$

$= 2x+0 = \boxed{2x}$

b) $y = -\frac{1}{2x}$

$y' = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \left(-\frac{1}{2(x+h)} + \frac{1}{2x}\right) \frac{1}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cancel{-x} + (\cancel{x}+h)}{2x(x+h)}\right]$

$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}}{2x(x+h)}\right]$

$= \lim_{h \to 0} \frac{1}{2x(x+h)}$

$= \frac{1}{2x(x+0)} = \boxed{\frac{1}{2x^2}}$

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3) Find the derivative of $y = 2x^2 - 5x$. After finding the derivative, find the slope of the tangent to the curve at $x = 3$ and then the equation of this tangent line. **(5 marks)**

$y' = 4x - 5$

$y'(3) = 4(3) - 5 = 12 - 5 = \boxed{7}$

$y = 7x + b$ pt: $(3, 2(3)^2 - 5(3)) = (3, 3)$ $18 - 15$

so find b: $3 = 7(3) + b$

$3 - 21 = b$

$b = -18$

$\boxed{y = 7x - 18}$

4) Find the derivative of each function below and simplify the result (show all your work for full marks): **(3 marks each)**

a) $y = (2x + 1)(x^2 - 5x + 1)$

$y' = (2)(x^2 - 5x + 1) + (2x + 1)(2x - 5)$

$= 2x^2 - 10x + 2 + 4x^2 - 10x + 2x - 5$

$= \boxed{6x^2 - 18x - 3} = 3(2x^2 - 6x - 1)$

b) $y = x + \pi - \frac{5}{\sqrt{x}} = x + \pi - 5x^{-1/2}$

$y' = \boxed{1 + \frac{5}{2x^{3/2}}}$

c) $y = \frac{12x^2 - 1}{(3x-5)^2}$

$y' = \frac{(24x)(3x-5)^2 - (12x^2-1)2(3x-5)(3)}{(3x-5)^2}$

$= \frac{6(3x-5)[4x(3x-5) - (12x^2-1)]}{((3x-5)^2)^2} = \boxed{\frac{6(1-20x)}{(3x-5)^3}}$

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$$= \left(2x+x^{3/4}\right)^{1/2}$$

d) $y = \sqrt{2x + x^{\frac{3}{4}}}$

$$y' = \frac{1}{2}\left(2x+x^{3/4}\right)^{-1/2}\left(2+\frac{3}{4}x^{-1/4}\right)$$

$$= \frac{2+\frac{3}{4x^{1/4}}}{2\sqrt{2x+x^{3/4}}} = \frac{1}{2\sqrt{2x+x^{3/4}}} \cdot \frac{8x^{1/4}+3}{4x^{1/4}} = \boxed{\frac{8x^{1/4}+3}{8x^{1/4}\sqrt{2x+x^{3/4}}}}$$

5) Sketch the graph of $y = \frac{x^2-1}{x}$. **(8 marks)**

a. Find the domain (1 mark)

$D_y = \mathbb{R} \setminus \{0\}$

b. Find the x and y intercepts and state any vertical or horizontal asymptote (2 marks)

x-int: $[y=0]$

$$0 = \frac{(x-1)(x+1)}{x}$$

$\boxed{x=1}$, $\boxed{x=-1}$

y-int $[x=0]$

impossible

$\boxed{\text{no y int}}$

VA:

$\boxed{x=0}$

$$\lim_{x\to 0^-} \frac{x^2-1}{x} \frac{\ominus}{\ominus} = +\infty$$

$$\lim_{x\to 0^+} \frac{x^2-1}{x} \frac{\ominus}{+} = -\infty$$

HA:

$$\lim_{x\to\infty} \frac{x^2-1}{x}$$

$\boxed{\text{No HA}}$

degree num > degree deno

c. Compute the derivative and find the critical points (1 mark)

$$y' = \frac{2x\cdot x-(x^2-1)}{x^2}$$

$$= \frac{2x^2-x^2+1}{x^2}$$

$$= \frac{x^2+1}{x^2}$$

C.P.: $\frac{x^2+1}{x^2} = 0$

never

$\boxed{\text{No c.P.}}$

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d. Draw the table of increase and decrease of $f(x)$ and find the maximum and minimum points. (1 mark)

	$-\infty$ $\quad x=-1$	0	$x=1 \quad \infty$
$y'=\frac{x^2+1}{x^2}$	$\oplus$	‖	$\oplus$
y	↗	‖	↗

increase: $x \in (-\infty, 0) \cup (0, \infty)$

decrease: never

local max; no

min: no

e. Compute the second derivative and find the inflection points (1 mark)

$$y'' = \frac{2x \cdot x^2 - (x^2+1)(2x)}{x^4} \qquad y' = \frac{-2}{x^3}$$

$$= \frac{2x(\cancel{x^2} - \cancel{x^2} - 1)}{x^4}$$

$\underline{IP}$: $0 = \frac{-2}{x^3}$

never

$\boxed{\text{no IP}}$

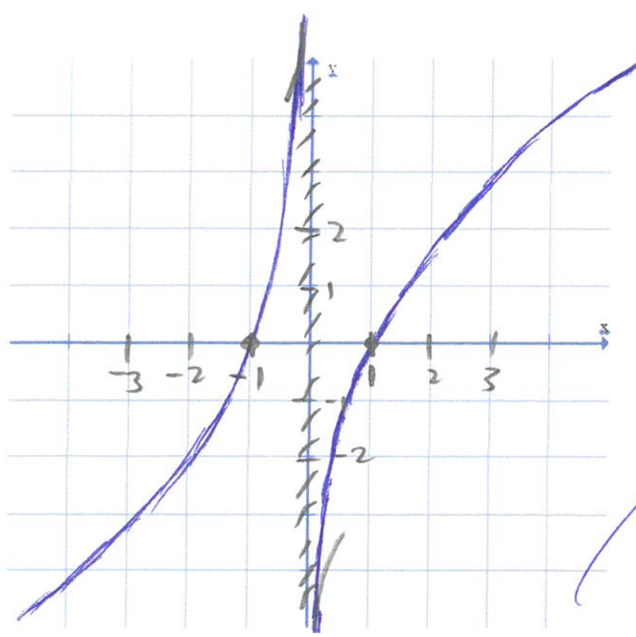
f. Draw the table of concavity (1 mark)

	$-\infty$ $\quad x=-1$	0	$x=1 \quad \infty$
$y''=\frac{-2}{x^3}$	$\frac{\ominus}{\ominus} > 0$	‖	$\frac{\ominus}{\oplus} < 0$
y	∪	‖	∩

concave up: $x \in (-\infty, 0)$

concave down: $x \in (0, \infty)$

g. Sketch the graph of $f(x)$ (1 mark)



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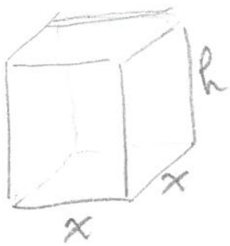
6) You want to wrap ~~of~~ a gift in a box that has a square base. The volume of the box needs to be 250 cm^3. How much wrapping paper do you need in order to wrap the box? (Hint: Find the dimensions of the box that minimize the surface area) **(5 marks)**



$$\boxed{V = 250cm^3 = x^2h}$$

Minimize $SA = 2x^2 + 4xh$

From volume formula: $h = \dfrac{250}{x^2}$

so $SA = 2x^2 + 4x\left(\dfrac{250}{x^2}\right) = 2x^2 + \dfrac{1000}{x}$

and $SA' = 4x - \dfrac{1000}{x^2}$ and to find critical pt

$0 = \dfrac{4x^3 - 1000}{x^2} \Rightarrow 4x^3 - 1000 = 0$

$x^3 = \dfrac{1000}{4} = 250$

and $\boxed{x = \sqrt[3]{250} = 6.2996cm}$

and $\boxed{h = \dfrac{250}{x^2} = \dfrac{250}{\underbrace{(\sqrt[3]{250})^2}_{39.68496}} \doteq 6.2996cm}$

$SA = 2x^2 + 4xh = 2(6.2996)^2 + 4(6.2996)(6.2996) = 6(6.2996)^2$

$$\boxed{SA = 238.1098cm^2}$$