Lecture Guides

For most of the content, you should understand it both graphically and algebraically.

1 Lecture 1

1.1 General inverse function

In order for a function f to have its inverse f^{-1}, f must be a "one-to-one" function (P 56). That is, its graph must pass through both vertical line test and horizontal line test. Or, if you try to solve x from $y = f(x)$ in terms of y, you should have at most 1 solution.

If a function is invertible, it will have two equivalent forms: $y = f(x)$ or $x = f^{-1}(y)$. Depending on different cases, one of them will be better to understand. For example, $y = 2^x$ or $x = \log_2 y$.

> Key contents are:
>
> - Obtaining the inverse function from a given function if it exists. (P 58)
> - Logarithm function. Calculation laws of logarithm function, logarithm equations. (P 60)
> - Trig function and their inverses. Trig identities/formulas Specifically, calculations involving their inverses. (P 64)

1.2 Intuitive idea of limit

Limit is more of a dynamic idea that is your "prediction" based on the observation of "neighboring" data. Since we only consider cases in 1D, we only have left and right neighbors. Given a point a, your prediction of $f(a)$ by only observing the behavior of f at data collected on the left of a (i.e. $x < a$) gives the left limit $\lim_{x\to a^-} f(x)$. The right limit $\lim_{x\to a^+} f(x)$ is defined in a similar way (P88). You should be aware that limits at a has NOTHING to do with the actual value at that point (P 84).

Here is a little summary if you get confused about all these definitions:

Theorem 1 $\lim_{x\to a} f(x)$ *is well defined if and only if* $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = c$ *where* c *should be a finite number. In this case,* $\lim_{x\to a} f(x) = c$.

If further $\lim_{x\to a} f(x) = f(a)$, *then function* f *is said to be continuous at* $x = a$.

If $\lim_{x\to a^-} f(x) = f(a)$, *then function* f *is said to be continuous from the left at* $x = a$.

If $\lim_{x\to a^+} f(x) = f(a)$, *then function* f *is said to be continuous from the right at* $x = a$.

> Key contents:
>
> - Find limits of different kinds from a graph. (Lots of exercises on P 92)
> - Find the interval where a given function is continuous. (P 122)

2 Lecture 2

2.1 Limit calculation

Almost all functions you commonly see in our class is continuous in their DOMAIN. They are polynomials, rational functions, power functions, trigonometry functions, exponential functions and logarithm functions (P 120). It is lucky that operations like add/subtract, multiplication/division and composition will "preserve" limit/continuity (again, x must be in the DOMAIN) if functions involved are ALL continuous (P 95, P 117 and P 121). This brings the convenience when we start to calculate some limits. Generally, a "3-stage" strategy can be used for limit calculation.

1 Directly plug in to see if you got a valid output.

2 If not, try to see if you can "simplify" the given expression. Classical techniques includes: factor, rationalization,... (P 98 and P 121)

3 In rare cases of our class, you may use some theorems to help find the limit. Squeeze Theorem is commonly used. (P101)

Limits at infinity and limits with values at infinity are special limits that worth being paid attention to. Two common questions are finding possible vertical asymptotes (VA) and horizontal asymptotes (HA).

- $x = a$ is a VA if ONE of the following is true (P 90):
  $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$ or $\lim_{x\to a^+} f(x) = -\infty$
- $y = c$ is a HA if ONE of the following is true (P 128):
  $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$

Key contents:

- Calculations of limits of different kinds, both graphically and algebraically. (P 98 - P 101)
- Using the correct mathematical notation when you present your answers. Please try to mimic those done in examples of the book.
- Remember some common limits. (P 129 - P 132)

3 Lecture 3

3.1 Continuity

We have talked about continuity's definition in the previous lecture. The reason why we study continuous functions is that they have many nice properties. We have used the continuous property when we calculated some limits by directly plugging in the x value (P119-P121).

Another one of these important properties is Intermediate Value Theorem (IVT, P122). Before you apply this theorem, you have to check:

- function f should be CONTINOUS on a CLOSED interval
- $f(a)$ and $f(b)$ should have different sign, i.e. one positive and the other negative.

Key contents:

- Learn to use IVT when necessary. Learn to write a solid mathematical proof (P123).
- Really understand the DEFINITION of continuity and be able to use it for questions like "find the domain where f is continuous" or "What value of certain parameter makes a piecewise function continuous" ... (P122 and exercises.)

3.2 Intro to Derivative

The first derivative of function f at x is defined as a limit (P152):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

If this limit exists and is a finite number, then we call f is differentiable at x, otherwise, we say f is not differentiable at x. There are some other notations which have the same meaning, for example, $\frac{df}{dx}$ (P 155).

$f'(x)$ becomes a function as x varies, we call $f'(x)$ the first derivative of f. Its geometrical meaning is the slope of the tangent line at x on the graph of f. Its physical meaning describes the rate of change of quantity f, for example the first derivative of displacement function gives the velocity, whose derivative then provides the acceleration function.

Key contents:

- Calculation of the first derivative from its definition as a limit. (P 154)
- Understand the physical meaning of first/second derivatives. (P 159)
- Understand the geometrical meaning of the first derivative.

4 Lecture 4

4.1 Differentiability and continuity

Continuity is a "broader" definition that contains differentiability. That is, a differentiable function is surely continuous, but a continuous function may not be differentiable somewhere (P156). Actually, there is a function that is continuous everywhere on $\mathcal{R}$ but nowhere differentiable! Their relation can be visualized in the following graph:

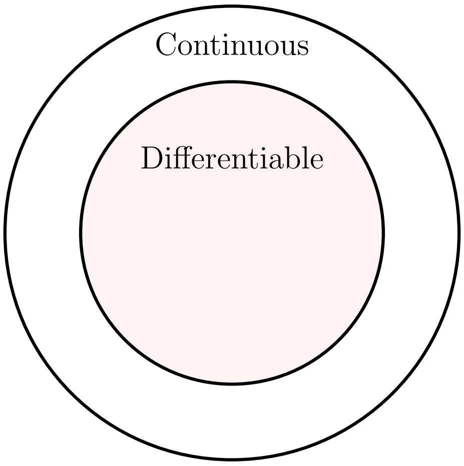

Figure 1: Relation between continuous functions and differentiable functions.

Key contents:

- Can identify points where a function is not differentiable. (P158)

4.2 Rules of taking derivatives

As some of you request in class, one proof of the product rule is given as below:

$$
\begin{aligned}
[fg]' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} && (2)\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} && (3)\\
&= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h} && (4)\\
&= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && (5)\\
&= f'(x)g(x) + f(x)g'(x) && (6)
\end{aligned}
$$

Notice that the key technique is inserting $-f(x)g(x+h) + f(x)g(x+h)$ in the second step to help break down the expression into two parts. You can distribution the limit to each factor is because we already assume the limit of each factor exists. Now you can try to give a proof of yours for quotient rule.

Again, the following rules works if only all involved derivatives exist (P 187).

Key contents:

Constant: $C' = 0$;

Power: $(x^{\alpha})' = \alpha x^{\alpha-1}$;

Constant multiple: $(Cf(x))' = Cf'(x)$;

Exponential: $(a^x)' = a^x \ln a$, $a = e$ gives you the result with natural base;

Logarithm: $\log_a(x)' = \frac{1}{x \log_a(x)}$, $a = e$ gives you the result with natural base;

Sum/difference: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$;

Product:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x);$$

Quotient:

$$[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - f(x)'g(x)}{g^2(x)}$$

5 Lecture 5

5.1 Derivative of Trig functions

Calculations involving trig functions can be complicated and technical. You should prepare yourself by remembering some basic identities/formulas, such as $\sin^2 x + \cos^2 x = 1$. There are also some useful results you may choose to remember.

$$\lim_{x \to 0} \frac{\sin x}{x} = 0; \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

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Key contents:

- Remember the derivative formulas. (P 193)
- Be able to calculate some basic trig limits.(P 195)

5.2 Chain Rule

This is a VERY IMPORTANT topic, you'll find that many contents in our future course rely on it. The sooner you can get yourself familiar with chain rule, the easier when you are doing calculations later. Chain rule deals with the situation when one needs to take the derivative of a composite function. The statement of chain rule is simple (P 198):

$$[f \circ g]' = f'[g(x)]g'(x). \tag{7}$$

It is basically a chain of multiplication of function derivatives. You should be careful with what to be evaluated at each of the 'chain'. That is the question: if you know certain function is composited as $f \circ g \circ h$, the derivative of it is

$$f'[?]g'[?]h'[?].$$

What to put to each question mark? The correct answer will be:

$$f'\{g[h(x)]\}g'[h(x)]h'[x].$$

Key contents:

- Be able to apply chain rule in various of derivative calculations. (P 201 -203) (More exercises...☹)

6 Lecture 6

The advanced techniques mentioned below along with previous ones can be combined in different order. They merely provide an option, you always have the freedom to choose which one to use and give a simpler calculation than the book does.

6.1 Implicit Derivative

An implicit form is a form like $f(x, y) = 0$ where f is some function depending on both x and y. An easy example is the equations for a circle with origin at $(0, 0)$ and radius r: $x^2 + y^2 = r^2$.

There are two important steps you should be careful when applying implicit derivation. First, answer questions: *What symbol is the independent variable? What are other variables? What symbols that are constants?* In the previous example, x is the independent variable, y is a variable that depends on x and r represent the constant. Second, when you start to take derivatives, do not forget to apply **chain rule**. That is, for symbols that you identify as a dependent variable, do not forget to attach a "prime" of it. In the circle equation case, you should have $2x + 2yy' = 0$ instead of $2x + 2y = 0$!

Another technique that highly depends on implicit differentiation is "Inverse derivatives". You should know how to convert between the *inverse form* and *regular form* in the first place.

Key contents:

- Use this technique to take derivatives of implicit equations.
- Change inverse form to its regular form and then take derivatives implicitly. (P214)
- Remember formulas for derivatives of inverse trig functions. (P214)

6.2 Logarithm Derivative

Logarithm Derivative is most suitable if you want to take derivatives on a gigantic multiplication/quotient of exponentials, radical functions or power functions. There are basically three steps to apply it (P221). You should know two things in the first place: Derivative formulas for general log functions (P218) and properties of log expressions (P60).

Key contents:

- Be able to use this technique to calculate derivatives for some complicated functions.

7 Lecture 7

7.1 Related Rates

This section is an application of those techniques we have learned in previous sections. It depends heavily on implicit derivative. The KEY step (also maybe the most difficult step) is to extract an equation with variables from the problem description, after which you can then apply implicit derivative techniques to solve the problem. When you try to establish the equation, be careful about quantities that are variables and quantities that are constants (P247). Topics like this are usually difficult, You may use several examples to help understand the strategy for problems of this kind.

Key contents:

- Learn to solve problems of relative rates.
- Deepen your understandings of variables and their physical relations in a mathematical way.

7.2 Linear Approximation

This section gives a simple formula for estimating a function value at X using its *nearby* point a where $f(a)$ is easy to evaluate. The formula is

$$f(x) \approx L_a(x) = f(a) + f'(a)(x - a) \tag{8}$$

It also introduces a definition of *differentials*, which you can use in error estimation.

Key contents:

- Use the linear approximation formula to estimate function value. (P252)
- Understand the definition of differential and use it for estimation. (P255)

7.3 About the first exam

Our first exam is on Oct 11, Wednesday in class. NO calculates, notes, books, digital devices are allowed! Scrap paper is allowed.

Tips:

- Your DGD questions are best reviewing materials, make sure you understand how to solve those questions. Do some exercises to ensure that these things you understood do not just rest in your brain.
- Show your work. Leaving only the final answer is as bad as writing down nothing.
- Read the question description carefully. Some questions require specific method.
- Never hurt to do a double check after you finished.
- Maybe go to a party after the exam.

8 Lecture 8

8.1 differentials

The differential of a function $f(x)$ is defined as $df = f'(x)dx$. This quantity can be used to estimate corresponding errors of output $f(x)$ when the input x comes with certain amount of measurement error dx.

Key contents: Know how to calculate differentials from its definition and use it for error estimation. (P255)

8.2 Antiderivatives

You can view antiderivative as the "inverse" operation of taking derivatives. That is, if a function $F(x)$ satisfies $F(x)' = f(x)$ for given $f(x)$, $F(x)$ will be called the antiderivative of $f(x)$(P 350) and can be written as $\int f(x)dx = F(x)$. Not like taking derivatives which at most gives you one solution, you can get multiple solutions if you take antiderivatives(P351). It is essential if you can remember some particular antiderivatives (P 352).

An equation that involves derivatives of a function is called a differential equation, which itself can be a huge topic and is one of the foundations of modern science.

Key contents:

- The definition of antiderivative (P351).
- Remembering some antiderivative formula for common functions (P 352).
- Solve some easy differential equations (P353-P354).