CARLETON UNIVERSITY

FINAL EXAMINATION SOLUTIONS
MATH 2004 A, B, C, D
Fall 2013

DURATION: 3 HOURS

Department Name and Course Number: School of Mathematics and Statistics, MATH 2004 A, B, C, D.
Course Instructor(s): Dr. A.B. Mingarelli (Sect. A), Dr. R. Cova (Sect. B, C), Dr. E. Xhua (Sect. D)

AUTHORIZED MEMORANDA
NON-PROGRAMMABLE CALCULATOR PERMITTED.
MANY **BLANK SHEETS OF PAPER** FOR ROUGH WORK

This exam may be released to the Library and **MAY NOT BE TAKEN AWAY BY THE STUDENT**.

- Please verify that you are in possession of a SCANTRON FORM
- Please fill in your COURSE CODE (e.g., MATH 2004) and COURSE SECTION (e.g., A, B, C, D), as per above list of instructors; YOUR NAME and YOUR STUDENT NUMBER where required on the Scantron form AND on this examination.
- **The entire examination consists of 5 pages and two parts, A and B, and is marked out of a total of 80**, that is, 40% of your final mark. Part A consists of 12 multiple choice questions each worth 3 marks. **Please fill in only one answer on your Scantron sheets with a pencil** as there is only one answer to any given question. Circling two or more answers to any question invalidates that question (*i.e.,* you get 0 marks for that question). Part B consists of 6 traditional type questions (with explanations required), for a total of 44 marks for that section. **Both Part A and B must be submitted along with the scantron sheet, so do not detach nor unstaple this examination. Do not submit rough work.**

Print Name :

Student Number:

Section (either A, B, C, or D):

- **A1.** Find the slope of the tangent line to $r = 3\sin\theta$ at $\theta = \dfrac{\pi}{2}$.

  (a) 0 (b) $\sqrt{3}$ (c) 2 (d) 3

  Solution: (a)

- **A2.** Let θ be the angle between the two planes $x + 2y - 2z - 3 = 0$ and $2x + 2y - z - 5 = 0$. Find $\cos\theta$.

  (a) $\dfrac{4}{9}$ (b) $\dfrac{9}{4}$ (c) $\dfrac{8}{9}$ (d) $\dfrac{9}{8}$

  Solution: (c)

- **A3.** Find the area of the triangle with vertices $P(1,2,3)$, $Q(-3,0,1)$, and $R(2,4,5)$.

  (a) $6\sqrt{2}$ (b) $6\sqrt{3}$ (c) 12 (d) $3\sqrt{2}$

  Solution: (d)

Print Name and Student Number: ____________________

- **A4.** Given the line $L : \mathbf{r} = (1,2,3) + t(3,2,-1)$. Let Π be the plane through the point $(1,1,3)$ and perpendicular to the line L. Which of the following is a point on the plane Π ?

  (a) $(0,0,-1)$ (b) $(0,0,-2)$ (c) $(0,0,-3)$ (d) $(0,0,-4)$

  Solution: (b)

- **A5.** Let $w(x,y) = x^2 + y + 3xy^4$, where $x = \sin 2t, y = \cos t$. Use the Chain Rule to find the value of $\dfrac{\partial w}{\partial t}$ at $t = 0$:

  (a) 0 (b) 6 (c) 3 (d) 2

  Solution: (b)

- **A6.** A vector giving the direction in which $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z}$ increases most rapidly at the point (1,-1,2) is given by

  (a) $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ (b) $-\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$ (c) $2\mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{1}{2}\mathbf{k}$ (d) $-\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{4}\mathbf{k}$

  Solution: (d)

- **A7.** Which of the following are critical points of $h(x,y) = y^2 - 2y - y\sqrt{x} + \dfrac{x}{2}$ where x, y are real, $x \geq 0$.

  (a) $(4,2)$, $(0,0)$ (b) $(0,0)$ only (c) $(2,4)$, $(0,0)$ (d) $(4,2)$ only

  Solution: (a)

- **A8.** Evaluate the double integral $\displaystyle\iint_{\mathcal{R}} e^{-x} \sin y \, dx \, dy$, where $\mathcal{R} = \{(x,y) \mid 0 \leq x \leq \ln 2,\ \ 0 \leq y \leq \pi/2\}$.

  (a) $1 - e$ (b) $\ln 2 - 1$ (c) 0 (d) $\dfrac{1}{2}$

  Solution: (d)

- **A9.** Evaluate the triple integral $\displaystyle\iiint_{\mathcal{T}} xyz \ dz \, dy \, dx$ where $\mathcal{T}$ is the region

  $\mathcal{T} = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ \sqrt{x^2 + y^2} \leq z \leq 2\}$.

  (a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{3}{8}$ (d) $\dfrac{3}{2}$

  Solution: (c)

- **A10**. Which one of the following integrals is equal to the double integral $\displaystyle\iint_{\mathcal{R}} 2xy \ dA$, under the change of variables $x = 3u, y = 4v$ where $\mathcal{R}$ is the elliptical region $16x^2 + 9y^2 \leq 144$.

  (a) $\displaystyle\iint_{\mathcal{S}} 12uv \ dA$ (b) $\displaystyle\iint_{\mathcal{S}} 288uv \ dA$, (c) $\displaystyle\iint_{\mathcal{S}} 8(u^2 + v^2) \ dA$ (d) $\displaystyle\iint_{\mathcal{S}} 12u^2v^2 \ dA$

  where $\mathcal{S} = \{(u,v) : u^2 + v^2 \leq 1\}$.

  Solution: (b)

Print Name and Student Number: ______________________________

- **A11.** Evaluate the line integral $\int_{\mathcal{C}} (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$ where $\mathcal{C}$ is the straight line from the point $(1, 2)$ to $(3, 4)$.

(a) 236 (b) 0 (c) 188 (d) 48

Solution: (a)

- **A12.** Let $\mathbf{F}(x, y, z) = 7x\mathbf{i} - z\mathbf{k}$. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\; dS$ where $\mathcal{S}$ is the surface of the sphere $x^2 + y^2 + z^2 = 4$ and $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

(a) 24π (b) 16π (c) 0 (d) 64π

Solution: (d)

End of Part A

Print Name and Student Number: ____________________

PART B: Do All Four (6) Questions for a total of 44 marks out of a maximum of 80.
Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **B1.** [8 marks] Using the method of Lagrange multipliers determine the local extrema and saddle points (if any) of $f(x, y, z) = x + 2y - 2z$ subject to the constraint $x^2 + 2y^2 + 4z^2 = 1$.

  Solution: $\nabla f = \lambda \nabla g,\ g = x^2 + 2y^2 + 4z^2 \longrightarrow \lambda = \pm 1$. The critical points are then $CP1(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4})$, $CP2(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4})$. One finds that $f(CP1) = 2$ and $f(CP2) = -2$ are the local maximum and minimum, respectively.

- **B2.** [7 marks] By changing to polar coordinates evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2)\, dx\, dy$.

  Solution: $\int_0^{\pi/2} \int_0^1 \cos(r^2)\, r dr\, d\theta = \frac{\pi}{4} \sin(1)$.

Print Name and Student Number: ____________________

- **B3.** [7 marks] The parametric curve C in the plane is given by $x = 2t^2$, $y = 1 + t^3$, where $0 \le t \le 1$. Find the length of the curve C.

Solution: We have $x'(t) = 4t, y'(t) = 3t^2$. The length is

$$\begin{aligned} L &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^1 \sqrt{16t^2 + 9t^4}\, dt = \int_0^1 t\sqrt{16 + 9t^2}\, dt \\ &= \frac{1}{18}\int_{16}^{25} \sqrt{u}\, du, \quad u = 16 + 9t^2, \\ &= \frac{1}{27} u^{3/2}\Big|_{16}^{25} = \frac{61}{27}. \end{aligned}$$

- **B4.** [7 marks] Find the area of the region enclosed by the cardioid $r = 1 + \sin\theta$ and the circle $r = 3\sin\theta$.

Solution: The intersection points of the two curves are given by $1 + \sin\theta = 3\sin\theta$, from which we get $\sin\theta = 1/2$ or $\theta = \pi/6$ or $5\pi/6$. The area is then given by

$$\begin{aligned} A &= \int_\alpha^\beta \frac{1}{2}(r_0^2 - r_1^2)\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin^2\theta - (1 + \sin\theta)^2)\, d\theta \\ &= \pi. \end{aligned}$$

Print Name and Student Number: ______________________________

- **B5.** [7 marks] Let $\mathbf{F}(x,y,z) = \dfrac{1}{yz}\,\mathbf{i} - \dfrac{x}{y^2 z}\,\mathbf{j} - \dfrac{x}{yz^2}\,\mathbf{k}$.

a) Compute curl $\mathbf{F}$.
b) Find a scalar field $f(x,y,z)$ such that $\nabla f = \mathbf{F}$. If no such scalar field exists, explain why this is not possible.

Solution:

a) curl $\mathbf{F} = \mathbf{0}$.

b) $f(x,y,z) = \dfrac{x}{yz}$.

- **B6.** [8 marks] Let $\mathbf{F}(x,y,z) = -5y\,\mathbf{i} + 4x\,\mathbf{j} + z\,\mathbf{k}$ and let $\mathcal{S}$ be the surface bounded by the circle $x^2 + y^2 = 4$ and $z = 1$. Evaluate the surface integral $\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS$ using any method. Here $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

Solution:

Direct calculation gives that curl $\mathbf{F} = 9\mathbf{k}$. Inn additon, because the normal lies on the plane $z = 1$ it is clear that $\mathbf{n} = \mathbf{k}$. Hence curl $\mathbf{F} \cdot \mathbf{n} = 9$. Next the region $\mathcal{R}$ in the $uv-$plane is a circle of radius 2. It follows that

$$\begin{aligned}
\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS &= 9 \iint_{\mathcal{R}} du\, dv \\
&= 9 \cdot \text{Area of } \mathcal{R} \\
&= 9\pi 2^2 = 36\pi.
\end{aligned}$$