## Winter 2017 MATH 135 Midterm Solutions

1. Is $((P \implies Q) \implies R) \equiv ((\neg P \implies R) \wedge (Q \implies R))$? Prove that your answer is correct.

   **Solution:** We complete a truth table:

| $P$ | $Q$ | $R$ | $P \implies Q$ | $(P \implies Q) \implies R$ | $\neg P \implies R$ | $Q \implies R$ | $(\neg P \implies R) \wedge (Q \implies R)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | **T** | **T** | **T** | **T** | **T** |
| $T$ | $T$ | $F$ | **T** | **F** | **T** | **F** | **F** |
| $T$ | $F$ | $T$ | **F** | **T** | **T** | **T** | **T** |
| $T$ | $F$ | $F$ | **F** | **T** | **T** | **T** | **T** |
| $F$ | $T$ | $T$ | **T** | **T** | **T** | **T** | **T** |
| $F$ | $T$ | $F$ | **T** | **F** | **F** | **F** | **F** |
| $F$ | $F$ | $T$ | **T** | **T** | **T** | **T** | **T** |
| $F$ | $F$ | $F$ | **T** | **F** | **F** | **T** | **F** |

   Since $(P \implies Q) \implies R$ and $(\neg P \implies R) \wedge (Q \implies R)$ always have the same truth value, they are logically equivalent.

   **Alternate Solution:** We use a chain of logical equivalences to show $(P \implies Q) \implies R$ and $(\neg P \implies R) \wedge (Q \implies R)$ are logically equivalent.

$$
\begin{aligned}
(P \implies Q) \implies R &\equiv ((\neg P) \vee Q) \implies R \\
&\equiv (\neg((\neg P) \vee Q)) \vee R \\
&\equiv (P \wedge (\neg Q)) \vee R && \text{by } \textit{De Morgan's Laws (DML)} \\
&\equiv (P \vee R) \wedge ((\neg Q) \vee R) && \text{by logical Distributive Laws} \\
&\equiv (\neg P \implies Q) \wedge (Q \implies R)
\end{aligned}
$$

2. A proof and a disproof of the same statement are given below. For each of these arguments, indicate whether or not it has a fundamental error. If it does, explain what this error is. **[2 marks]**

   *Statement:* For all $x \in \mathbb{Z}$, $3 \mid (x^3 - x)$.

   (a) *Proof:* Suppose by way of contradiction that there exists $x \in \mathbb{Z}$ such that $3 \nmid (x^3 - x)$. When $x = 2$, $x^3 - x = 6$, so $3 \mid (x^3 - x)$. This is a contradiction.

   (b) *Disproof:* We will prove there exists $x \in \mathbb{Z}$ such that $3 \nmid (x^3 - x)$. Suppose by way of contradiction that $3 \mid (x^3 - x)$ for all $x \in \mathbb{Z}$. Then $3 \mid x^3$ and $3 \mid x$ by *Divisibility of Integer Combinations (DIC)*. When $x = 2$, this implies $3 \mid 2$. This is a contradiction.

   **Solution:**

   (a) There is a fundamental error. The existence of $x$ such that $3 \nmid (x^3 - x)$ does not allow us to assume $x = 2$. Considering only this case is tantamount to proving the statement by example.

   (b) There is a fundamental error. The use of DIC is incorrect. The implication "if $a \mid (b + c)$, then $a \mid b$ and $a \mid c$" is not true in general.

3. (a) Negate the statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + 3y = 2$. **[5 marks]**
(Do not use negative words such as "not" or the symbol $\neg$, but negative symbols like $\nmid$ are okay.)

(b) Consider the statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + 3y = 2$.
Is the statement true or false?
Prove or disprove the statement.

(c) Consider the statement: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x^2 + 3y = 2$.
Is the statement true or false?
Prove or disprove the statement.

**Solution:**

(a) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + 3y \neq 2$

(b) The statement is true.
To prove the statement, let $x \in \mathbb{R}$ and consider $y = \frac{2-x^2}{3}$. Then $y \in \mathbb{R}$ and

$$x^2 + 3y = x^2 + 3\left(\frac{2-x^2}{3}\right) = x^2 + 2 - x^2 = 2.$$

(c) The statement is false.
To disprove the statement, take $y = 1$ as a counterexample. Then $x^2 + 3y = 2$ is equivalent to $x^2 = -1$ and we know $x^2 \geq 0$ for all real numbers $x$ so no real $x$ satisfies $x^2 + 3y = 2$ when $y = 1$.

4. Let $a$ and $b$ be rational numbers. Consider the implication $S$: **[6 marks]**

If $a < b$, then there exists a rational number $c$ such that $a < c < b$.

In your answers to parts (a) to (e) below, do not use negative words such as "not" or the symbol $\neg$, and also do not use negative symbols like $\nmid$.

(a) State the hypothesis of $S$.

(b) State the conclusion of $S$.

(c) State the converse of $S$.

(d) State the contrapositive of $S$.

(e) State the negation of $S$.

(f) Prove $S$.

**Solution:**

(a) $a < b$

(b) there exists a rational number $c$ such that $a < c < b$

(c) If there exists a rational number $c$ such that $a < c < b$, then $a < b$.

(d) If for all rational numbers $c$, $c \leq a$ or $c \geq b$, then $a \geq b$.

(e) $a < b$ and for all rational numbers $c$, $(c \leq a$ or $c \geq b)$

(f) Let $a$ and $b$ be rational numbers where $a < b$. Consider $c = \frac{a+b}{2}$. Then $c$ is rational because $a$ and $b$ are rational and 2 is a non-zero integer. Also,

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b$$

as required.

5. Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$. That is, prove *Transitivity of Divisibility (TD).* **[4 marks]**

   **Solution:** Suppose $a \mid b$ and $b \mid c$ for integers $a$, $b$ and $c$. Then there exist integers such that $b = ak$ and $c = b\ell$. Substituting gives $c = (k\ell)a$. Now $k\ell$ is an integer because $k$ and $\ell$ are integers, so $a \mid c$.

6. Let $x \in \mathbb{R}$ such that $\cos x \neq 0$. Prove that $\cos^2 x - \sin^2 x = 0$ if and only if $|\tan x| = 1$. **[3 marks]**

   **Solution 1:** We begin by assuming that $\cos^2 x - \sin^2 x = 0$. Thus $(\cos x - \sin x)(\cos x + \sin x) = 0$ and so $\cos x = \sin x$ or $\cos x = -\sin x$. Since $\cos x \neq 0$, we can divide both sides of both equations by $\cos x$ to obtain that $1 = \tan x$ or $1 = -\tan x$. Thus $\tan x = \pm 1$ and so $|\tan x| = 1$.

   Conversely, assume $|\tan x| = 1$. Thus $\left|\frac{\sin x}{\cos x}\right| = 1$ which is equivalent to $\frac{|\sin x|}{|\cos x|} = 1$. We can multiply both sides by $|\cos x|$ to obtain $|\sin x| = |\cos x|$. Squaring both sides we obtain $|\sin x|^2 = |\cos x|^2$ which is equivalent to $|\sin^2 x| = |\cos^2 x|$. Since $\sin^2 x, \cos^2 x \geq 0$, then we obtain that $\sin^2 x = \cos^2 x$. Thus $\cos^2 x - \sin^2 x = 0$.

   **Solution 2:** We will use a series of equivalent statements.

$$\begin{aligned} & \cos^2 x - \sin^2 x = 0 \\ \iff & (\cos x - \sin x)(\cos x + \sin x) = 0 \\ \iff & \cos x = \sin x \text{ or } \cos x = -\sin x \\ \iff & 1 = \tan x \text{ or } 1 = -\tan x \quad (\text{since } \cos x \neq 0) \\ \iff & |\tan x| = 1 \end{aligned}$$

   Therefore, $\cos^2 x - \sin^2 x = 0$ if and only if $|\tan x| = 1$.

7. Let $a, b \in \mathbb{Z}$. Prove that if $2a^2 - 3ab + 4a - 6b$ is odd, then $a$ is odd. **[3 marks]**

   **Solution 1:** Let $a, b \in \mathbb{Z}$. We will prove the contrapositive. Suppose $a = 2k$ for some integer $k$. Then $2a^2 - 3ab + 4a - 6b = 2(a^2 - 3kb + 2a - 3b)$. Since $a$, $b$ and $k$ are integers, $a^2 - 3kb + 2a - 3b$ is also an integer. Therefore $2a^2 - 3ab + 4a - 6b$ is even.

   **Solution 2:** Let $a, b \in \mathbb{Z}$. We will prove the contrapositive. Suppose $a$ is even. Then since the product of an even integer and any even integer is even, we see that $2a^2$, $-3ab$ and $4a$ are even. We also know $6b = 2(3b)$ is even. Hence $2a^2 - 3ab + 4a - 6b$ is even because the sum of any number of even integers is even.

8. Let $A, B$ and $C$ be sets. Prove that if $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B \subseteq C$. **[3 marks]**
   (For this question it is especially important that you write a very clear and formal proof.)

   **Solution 1:** Let $A, B$ and $C$ be sets where $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Let $x \in B$. Then $x \in A \cup B$ by definition of union. Hence, $x \in A \cup C$ by assumption. Thus $x \in A$ or $x \in C$. If $x \in C$, we are done so suppose $x \in A$. Then $x \in A \cap B$ by definition of intersection. Hence $x \in A \cap C$ by assumption. Thus $x \in C$. In either case, $x \in C$. Hence $B \subseteq C$ as required.

   **Solution 2:** Let $A, B$ and $C$ be sets. We will prove the contrapositive. Suppose there is an element $x \in B$ but $x \notin C$. If $x \in A$, then $x \in A \cap B$ but $x \notin C$ by definition of intersection. That is, $A \cap B \neq A \cap C$. On the other hand, if $x \notin A$, then $x \in A \cup B$ but $x \notin A \cup C$ by definition of union. That is, $A \cap B \neq A \cap C$ is also true in this case.

   **Solution 3:** Suppose by way of contradiction that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ and there exists some $x \in B$ with $x \notin C$ for sets $A, B$ and $C$. Now $x \notin A \cap C$ by definition of intersection. Hence $x \notin A \cap B$ by assumption. This means $x \notin A$ by definition of intersection. On the other hand, $x \in A \cup B$ by definition of union. Hence $x \in A \cup C$ by assumption. However, this means $x \notin C$ implies $x \in A$ by definition of union. We have that $x \in A$ and $x \notin A$. This is a contradiction.

9. Let the sequence $\{a_n\}$ be defined by $a_1 = 2$, $a_2 = 1$, and $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ for all integers $n \geq 1$. **[4 marks]**

**Solution:**

Let $P(n)$ be the statement to be proven.

*Base Case*: When $n = 1$, $a_1 = 2$ and $3^n = 3$ so since $2 \leq 3$, $P(1)$ is true. When $n = 2$, $a_1 = 1$ and $3^n = 9$ so since $1 \leq 9$, $P(2)$ is true.

*Inductive Hypothesis*: We assume that the statement $P(i)$ is true for all integers $i$ satisfying $1 \leq i \leq k$ and some integer $k \geq 2$.

*Inductive Conclusion*: When $n = k + 1$,

$$\begin{aligned} a_n &= 2a_k + 2a_{k-1} \quad \text{by definition} \\ &\leq 2 \cdot 3^k + 2 \cdot 3^{k-1} \quad \text{by the inductive hypothesis} \\ &= 2 \cdot 3 \cdot 3^{k-1} + 2 \cdot 3^{k-1} \\ &= (6 + 2) \cdot 3^{k-1} \\ &= 8 \cdot 3^{k-1} \\ &\leq 3^2 \cdot 3^{k-1} \\ &= 3^{k+1} \end{aligned}$$

Therefore, by the *Principle of Srong Induction (POSI)*, $P(n)$ is true for all natural numbers $n$.

10. Use induction to prove that $3 \mid (5^n + 2^{n+1})$ for every natural number $n$. **[4 marks]**

**Solution:**

Let $P(n)$ be the statement to be proven.

*Base Case*: When $n = 1$, $5^n + 2^{n+1} = 9$ and $3 \mid 9$ so $P(1)$ is true.

*Inductive Hypothesis*: We assume that the statement $P(k)$ is true for some natural number $k$. That is, assume $3 \mid (5^k + 2^{k+1})$.

*Inductive Conclusion*: When $n = k + 1$,

$$\begin{aligned} 5^n + 2^{n+1} &= 5^{k+1} + 2^{k+2} \\ &= 5 \cdot 5^k + 2 \cdot 2^{k+1} \\ &= (3 + 2)5^k + 2 \cdot 2^{k+1} \\ &= 2 \cdot (5^k + 2^{k+1}) + 3 \cdot 5^k. \end{aligned}$$

Now $3 \mid 3 \cdot 5^k$ because $5^k$ is an integer, and $3 \mid (5^k + 2^{k+1})$ by the inductive hypothesis. Thus by *Divisibility of Integer Combinations (DIC)*, $3 \mid ((5^k + 2^{k+1})x + 3 \cdot 5^k y)$ where $x = 2$ and $y = 1$. Therefore $3 \mid (5^n + 2^{n+1})$ and $P(n)$ is true for all $n \in \mathbb{N}$ by the *Principle of Mathematical Induction (POMI)*.

11. A *prime* is an integer $n > 1$ for which its only positive divisors are 1 and $n$.
Prove the following statement. **[3 marks]**

There does not exist a natural number $k$ such that for all natural numbers $x$, $x^2 + 3x + k$ is prime.

**Solution:** By way of contradiction, suppose that $k$ is a natural number such that for all natural numbers $x$, $x^2 + 3x + k$ is prime. We consider two cases.

When $k = 1$, $x^2 + 3x + k = x^2 + 3x + 1$ and when $x = 6$, $x^2 + 3x + 1 = 55$ which is not prime.

When $k > 1$, consider when $x = k$, then $x^2 + 3x + k = k^2 + 3k + k = k(k + 4)$. Since $k > 1$, $k \geq 2$ and $k + 4 \geq 2$ and thus $x^2 + 3x + k$ is composite.

In either case we know that $x^2 + 3x + k$ is not prime for some natural number $x$. This is a contradiction completing the proof.