1. A) Determine the values of c and k that make the following function continuous. (Justify all reasoning)

$$f(x) = \begin{cases} 3x + k & x < 0 \quad ① \\ x^2 - 1 & 0 \le x \le 2 \quad ② \\ \sqrt{cx+3} & x > 2 \quad ③ \end{cases}$$

① is continuous everywhere as it is linear
② is continuous everywhere as it is quadratic
③ is continuous everwhere as long as $cx + 3 > 0$ (we check this later)

we check jump discontinuities @ $x=0$ and $x=2$

$\lim_{x \to 0^-} 3x+k = k$ $\quad \lim_{x\to 0^+} x^2-1 = -1$

$\therefore k=-1$ will make it cont at 0.

$\lim_{x\to 2^-} x^2-1 = 3$ $\quad \lim_{x\to 2^+} \sqrt{cx+3} = \sqrt{2c+3}$

$\therefore 3 = \sqrt{2c+3}$
$9 = 2c+3$ → $c=3$

$\therefore c=3$ makes cont @ 2

since there are no more jumps and $\sqrt{2x+3}$ will make $2x+3>0$ when $x>2$, f will be cont everywhere

B) Find the following limit $\lim_{x\to 0}\left(1+\frac{3}{x}\right)^{2x}$

$L = \lim_{x\to 0} \left(1+\frac{3}{x}\right)^{2x}$

$\ln L = \lim_{x\to 0} \ln\left(1+\frac{3}{x}\right)^{2x}$

$= \lim_{x\to 0} 2x \ln\left(1+\frac{3}{x}\right)$

$= 2\lim_{x\to 0} \dfrac{\ln\left(1+\frac{3}{x}\right)}{x^{-1}} \quad \frac{\infty}{\infty}$

$\therefore = 2\lim_{x\to 0} \dfrac{\frac{1}{1+3/x}\left(-3x^{-2}\right)}{-x^{-2}}$

$= 2\lim_{x\to 0} \dfrac{3}{1+3/x} = 2\left(\dfrac{3}{1+\infty}\right) = 0$

$\therefore \ln L = 0$
$L = e^0 = 1$

C) Find the following limit $\lim_{x\to\infty} \sqrt{2x+1} - \sqrt{2x+7}$

$\infty - \infty$ $\therefore$ more work.

$$= \lim_{x\to\infty} \frac{\sqrt{2x+1} - \sqrt{2x+7}}{1} \frac{(\sqrt{2x+1} + \sqrt{2x+7})}{(\sqrt{2x+1} + \sqrt{2x+7})}$$

$$= \lim_{x\to\infty} \frac{2x+1 - (2x+7)}{\sqrt{2x+1} + \sqrt{2x+7}}$$

$$= \frac{-6}{\infty + \infty} = 0$$

2. A) Find the derivatives of the following three functions:

$$f(x) = \ln(x)\,(\sin(x)), \qquad g(x) = \frac{2x+1}{x^2-1}, \qquad h(x) = \log_3(2e^{3x^2}+1)$$

$$f' = \frac{1}{x}\sin x + \cos x \ln x$$

$$g' = \frac{2(x^2-1) - 2x(2x+1)}{(x^2-1)^2}$$

$$= \frac{-2x^2 - 2x - 2}{(x^2-1)^2}$$

$$h' = \frac{1(2e^{3x^2})(6x)}{(2e^{3x^2}+1)\ln 3}$$

$$h' = \frac{12xe^{3x^2}}{(2e^{3x^2}+1)\ln 3}$$

B) Find the equation of the tangent line of $2e^y - x^2y = x$ at $(2,0)$

Derive implicitly

$$2e^y y' - 2xy - x^2 y' = 1$$

$$y'(2e^y - x^2) = 1 + 2xy$$

$$y' = \frac{1+2xy}{2e^y - x^2}$$

Sub in $(2,0)$ to find y'

$$y' = \frac{1+2(2)(0)}{2e^0 - 2^2} = -\frac{1}{2}$$

$\therefore y = -\frac{1}{2}x + b$ (Sub in $(2,0)$)

$0 = -\frac{1}{2}(2) + b$

$b = 1$ $\therefore y = \frac{-1}{2}x + 1$

C) Determine the linearization of $g(x) = \sqrt[3]{x}$ at $x = 27$ and use this equation to approximate the value of $\sqrt[3]{26}$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= 3 + \frac{1}{27}(x-27)$$

$f(a) = \sqrt[3]{27} = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(a) = \frac{1}{3}\frac{1}{\sqrt[3]{27}^2} = \frac{1}{27}$

$L(26) \approx \sqrt[3]{26}$ (as 26 is close to 27)

$$\therefore \sqrt[3]{26} \approx 3 + \frac{1}{27}(26-27)$$

$$\approx 3 - \frac{1}{27} \approx \frac{80}{27}$$

D) Using logarithmic differentiation, find the derivative of: $f(x) = \frac{e^{2x}(x^3-1)^3}{x^5(1-3x)^6}$

$$\ln y = \ln\left(\frac{e^{2x}(x^3-1)^3}{x^5(1-3x)^6}\right)$$

$$\ln y = 2x \cancel{\ln e} + 3\ln(x^3-1) - 5\ln x - 6\ln(1-3x)$$

Derive implicitly

$$\frac{1}{y}y' = 2 + \frac{9x^2}{x^3-1} - \frac{5}{x} + \frac{18}{1-3x}$$

$$\therefore y' = \left(2 + \frac{9x^2}{x^3-1} - \frac{5}{x} + \frac{18}{1-3x}\right)\frac{e^{2x}(x^3-1)^3}{x^5(1-3x)^6}$$

3. A) Consider a function $f(x) = 12\sqrt[3]{x^2}e^{\frac{1}{6}x}$ which has the derivative given as:

$$f'(x) = 2x^{-\frac{1}{3}}e^{\frac{1}{6}x}(4+x)$$

i) Determine the intervals which f is increasing and decreasing.

ii) Find all critical points and classify the points as local max, local min or neither.

Crit points $\Rightarrow$ when f' is undefined or $f' = 0$

$x^{-\frac{1}{3}} \Rightarrow x = 0$ is undefined, but $x = 0$ is in F

$\therefore$ a crit point @ $x = 0$

$e^{1/6x} \neq 0$ anywhere and is always defined

$4 + x = 0$ @ $x = -4$ $\therefore$ another crit point

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we note f has domain $(-\infty, \infty)$ as there are no square roots, ln's, etc..

$\therefore$ our table is

f'	$-\infty$	-5	-4	-1	0	1	∞
$2x^{-1/3}$		$-$		$-$		$+$	
$e^{1/6x}$		$+$		$+$		$+$	
$4+x$		$-$		$+$		$+$	
Total		$+ \nearrow$		$\searrow -$		$\nearrow +$	

$\therefore$ Increasing on $(-\infty, -4)$ and $(0, \infty)$

Decreasing on $(-4, 0)$

$\therefore$ Max @ $x = -4$

Min @ $x = 0$

B) Consider a function $g(x)=\frac{(x^2-1)}{2x^2-8}$ that has the following properties:

$D=(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

Intercepts: $(0,0.125)$, $(-1,0)$, and $(1,0)$

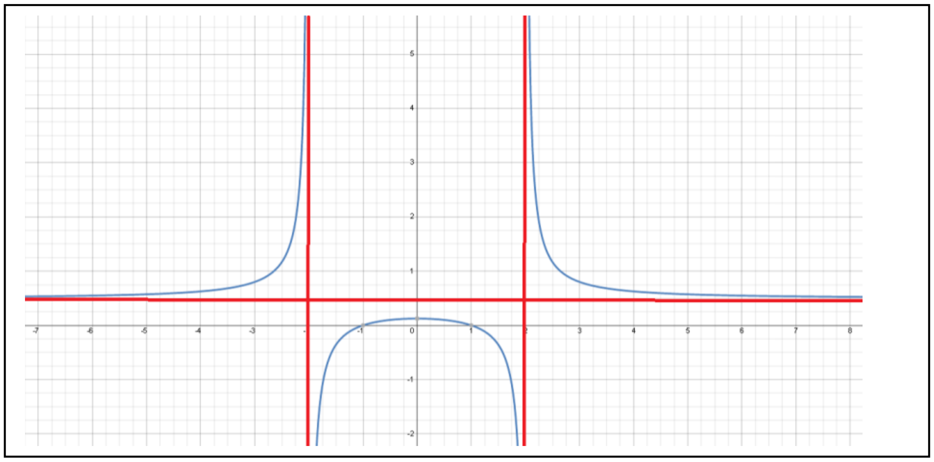
Critical points $x=0$

g is increasing on $(-\infty,-2)$ and $(-2,0)$; decreasing on $(0,2)$ and $(2,\infty)$

g is concave up $(-\infty,-2)$ and $(2,\infty)$; concave down on $(-2,2)$

Vertical asymptotes: x=2 and x=-2; horizontal asymptotes: y = 0.5

Provide a sketch of the function.



4. A) Given $f(x)=x^2-1$ and $g(x)=2x+7$, determine the points of intersection of the two functions.

$x^2-1=2x+7$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$\therefore x=4$

$x=-2$

$\therefore f(4)=16-1=15$

$f(-2)=4-1=3$

$\therefore$ POI are $(-2,3)$ and $(4,15)$

B) Determine the contained area between the two curves.

$$\left|\int_{-2}^{4} f(x)-g(x)\,dx\right| = \left|\int_{-2}^{4} x^2-2x-8\,dx\right| = \left|\frac{x^3}{3}-x^2-8x\right|_{-2}^{4}$$

$$=\left|\frac{64}{3}-16-32-\left(\frac{-8}{3}-4+16\right)\right| = \left|\frac{72}{3}-60\right| = |24-60| = |-36|$$

$\therefore$ Area $=36$ units^2

C) Let $H(x)$ be the antiderivative of $f(x)$, determine the function $H(x)$ given that the point $(-3,1)$ lies on the graph of $H(x)$.

$H(x)=\int x^2-1$

$=\frac{x^3}{3}-x+C$

Sub in $(-3,1)$

$1=\frac{-27}{3}+3+C$

$1=-6+C$

$\therefore C=7$

$\therefore H(x)=\frac{x^3}{3}-x+7$