Solution to Midterm Test 2 (version A)

MAT 1322D, Fall 2016
Total = 20 marks

Part I. Multiple-choice questions ($3 \times 4 = 12$ marks)

Version A1: CDAE
Version A2: DABD

1. Suppose Euler's method with step size $h = 0.05$ is used to estimate $y(0.1)$, where $y(t)$ is the solution to the initial-value problem $y' = (2t - 1)y$, $y(0) = 1$. Which one of the following values is closest to the result that you obtained?

Version A1: (A) 0.95; (B) 0.93; (C) 0.91; (D) 0.89; (E) 0.87.
Version A2: (A) 0.85; (B) 0.87; (C) 0.89; (D) 0.91; (E) 0.93.

Solution. The iteration formula is $y_{n+1} = y_n + h(2t_n - 1)y_n$ with $t_0 = 0$, and $y_0 = 1$.

n	t_n	y_n
0	0	1
1	0.05	$1 + 0.05 \times (2 \times 0 - 1) \times 1 = 0.95$
2	0.1	$0.95 + 0.05 \times (2 \times 0.05 - 1) \times 0.95 = 0.90725$

$y(0.1) \approx 0.91$.

2. Suppose salted water of concentration 5 g / m^3 is added to a reservoir of volume 1000 m^3 at a rate 2 m^3/ minute. Assume the water in the reservoir is well mixed and the same amount of mixed water is removed from the reservoir. Let $Q(t)$ be the quantity, in grams, of salt in the reservoir at time t. The differential equation that $Q(t)$ satisfies is

Version A1:

(A) $Q' = 10Q - 0.002Q^2$; (B) $Q' = 5 - 0.002Q$; (C) $Q' = 10 - 0.005Q$;
(D) $Q' = 10 - 0.002Q$; (E) $Q' = 5Q - 0.005Q^2$.

Version A2:

(A) $Q' = 10 - 0.002Q$; (B) $Q' = 5Q - 0.005Q^2$; (C) $Q' = 10Q - 0.002Q^2$;
(D) $Q' = 5 - 0.002Q$; (E) $Q' = 10 - 0.005Q$.

Solution. $r_{\text{in}} = 2 \times 5 = 10$. $r_{\text{out}} = 2 \times \dfrac{Q}{1000} = 0.002Q$. The equation is $Q' = 10 - 0.002Q$.

3. The sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n} - (-1)^n 3^{n+1}}{5^n}$ is

Version A1:

(A) $\frac{25}{8}$; (B) $\frac{55}{8}$; (C) $\frac{33}{5}$; (D) $\frac{35}{8}$; (E) $\frac{28}{5}$.

Version A2:

(A) $\frac{35}{8}$; (B) $\frac{25}{8}$; (C) $\frac{55}{8}$; (D) $\frac{28}{5}$; (E) $\frac{33}{5}$.

Solution. $\sum_{n=0}^{\infty}\frac{2^{2n}-(-1)^n3^{n+1}}{5^n}=\sum_{n=0}^{\infty}\frac{2^{2n}}{5^n}-\sum_{n=0}^{\infty}\frac{(-1)^n3^{n+1}}{5^n}$. Both series on the right-hand side are geometric series. The first series has first term 1, and common ratio $\frac{4}{5}$, and the second series has first term 3 and common ratio $-\frac{3}{5}$. The sum of the series is $S =$ $\frac{1}{1-4/5}-\frac{3}{1+3/5}=5-\frac{15}{8}=\frac{25}{8}$.

4. Consider series $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$. Which one of the following statements is true?

Note that, when $n \ge 1$, we have $2n < 2n + \sin^2 n < 3n$ and $2n^2 < 2n^2 + \cos^2 n < 3n^2$. Hence

$$\frac{\sqrt{2n}}{3n^2}<\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}<\frac{\sqrt{3n}}{2n^2} \text{ when } n \ge 1.$$

Version A1:

(A) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}>\frac{\sqrt{2n}}{3n^2}=\frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}}=\frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ diverges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

(B) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}>\frac{\sqrt{2n}}{3n^2}=\frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}}=\frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ converges.

(C) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}>\frac{\sqrt{2n}}{3n^2}=\frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}}=\frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

(D) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} < \frac{\sqrt{3n}}{2n^2} = \frac{\sqrt{3}}{2n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{3}}{2n^{3/2}} = \frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ diverges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

(E) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} < \frac{\sqrt{3n}}{2n^2} = \frac{\sqrt{3}}{2n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{3}}{2n^{3/2}} = \frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ converges.

Solution. (E) Since $\sum_{n=}^{\infty}\frac{1}{n^{3/2}}$ converges. (A) and (D) are false. When $\sin^2 n$ and $\cos^2 n$ are omitted, the behavior of this series is similar to the series $\sum_{n=1}^{\infty}\frac{\sqrt{2n}}{2n^2} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2}n^{3/2}}$, which is convergent. (C) is false. To show a series is convergent, we need to compare with a convergent series with greater terms. Then (B) is false.

Version A2:

(A) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} > \frac{\sqrt{2n}}{3n^2} = \frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}} = \frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ diverges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

(B) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} > \frac{\sqrt{2n}}{3n^2} = \frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}} = \frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ converges.

(C) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} > \frac{\sqrt{2n}}{3n^2} = \frac{\sqrt{2}}{3n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{2}}{3n^{3/2}} = \frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

(D) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} < \frac{\sqrt{3n}}{2n^2} = \frac{\sqrt{3}}{2n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{3}}{2n^{3/2}} = \frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ converges.

(E) Since $\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n} < \frac{\sqrt{3n}}{2n^2} = \frac{\sqrt{3}}{2n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{\sqrt{3}}{2n^{3/2}} = \frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ diverges, $\sum_{n=1}^{\infty}\frac{\sqrt{2n+\sin^2 n}}{2n^2+\cos^2 n}$ diverges.

Solution. (D) Since $\sum_{n=}^{\infty}\frac{1}{n^{3/2}}$ converges. (A) and (E) are false. When $\sin^2 n$ and $\cos^2 n$ are omitted, the behavior of this series is similar to the series $\sum_{n=1}^{\infty}\frac{\sqrt{2n}}{2n^2} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2}n^{3/2}}$, which is convergent. (C) is false. To show a series is convergent, we need to compare with a convergent series with greater terms. Then (B) is false.

Part II. Detailed Answer questions (8 marks)

5. (4 marks) (A1) Solve the initial-value problem $y' = (1 + y)(5 - y)$, $y(0) = -3$.

Solution. $\displaystyle\int \frac{1}{(1+y)(5-y)}\,dy = \int dt$.

Since $\displaystyle\frac{1}{(1+y)(5-y)} = \frac{1}{6}\left(\frac{1}{1+y} + \frac{1}{5-y}\right)$, $\displaystyle\frac{1}{6}\ln\left|\frac{1+y}{5-y}\right| = t + C$.

$\displaystyle\left|\frac{1+y}{5-y}\right| = K_1 e^{6t}$, where $K_1 = e^{6C} > 0$. $\displaystyle\frac{1+y}{5-y} = Ke^{6t}$, $K = \pm K_1 \neq 0$.

By the initial values condition, $K = -\frac{1}{4}$. $\displaystyle\frac{1+y}{5-y} = -\frac{1}{4}e^{6t}$. $4 + 4y = -5e^{6t} + ye^{6t}$.

$$y = \frac{5e^{6t} + 4}{e^{6t} - 4}.$$

(A2) Solve the initial-value problem $y' = (1 - y)(5 + y)$, $y(0) = 3$.

Solution. $\displaystyle\int \frac{1}{(1-y)(5+y)}\,dy = \int dt$.

Since $\displaystyle\frac{1}{(1-y)(5+y)} = \frac{1}{6}\left(\frac{1}{1-y} + \frac{1}{5+y}\right)$, $\displaystyle\frac{1}{6}\ln\left|\frac{5+y}{1-y}\right| = t + C$.

$\displaystyle\left|\frac{5+y}{1-y}\right| = K_1 e^{6t}$, where $K_1 = e^{6C} > 0$. $\displaystyle\frac{5+y}{1-y} = Ke^{6t}$, $K = \pm K_1 \neq 0$.

By the initial values condition, $K = -4$. $\displaystyle\frac{5+y}{1-y} = -4e^{6t}$. $5 + y = -4e^{6t} + 4ye^{6t}$.

$$y = \frac{4e^{6t} + 5}{4e^{6t} - 1}.$$

6. (2 × 2 = 4 marks) Determine whether each of the following series is convergent or divergent. **Justify your answer**.

Version A1:

a. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; b. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{3n+1}$.

Solution. a. Since function $y = \frac{1}{x \ln x}$ is positive, decreasing and continuous when $x \geq 2$, we can use the integral test. $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u} du = \lim_{b\to\infty} (\ln \ln b - \ln \ln 2) = \infty$. This series diverges.

b. Since $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0$, this series diverges.

Version A2:

a. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$; b. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{3n^2+1}$.

Solution. a. Since function $y = \frac{1}{x(\ln x)^2}$ is positive, decreasing and continuous when $x \geq 2$, we can use the integral test.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2} du = \lim_{b\to\infty} \left(-\frac{1}{\ln b} - \left(-\frac{1}{\ln 2} \right) \right) = \frac{1}{\ln 2}.$$

This series converges.

b. This is an alternating series. Since $\frac{n}{3n^2+1}$ is decreasing and $\lim_{n\to\infty} \frac{n}{3n^2+1} = 0$, this series converges.