ENGR 213
APPLIED ORDINARY DIFFERENTIAL EQUATIONS
Sample Midterm Examination II

Please attempt all Problems 1-3. They have equal value.
Materials allowed: non-programmable calculators.

Problem 1
Solve the differential equation:

$$y'' - 2y' + y = e^x \ln x, \quad x > 0$$

Problem 2
Find a second solution of the following differential equation:

$$2x^2 y'' + 3xy' - y = 0, \quad x > 0$$

knowing one solution $y_1(x) = \dfrac{1}{x}$.

Problem 3
Solve the following differential equation

$$y'' + 4y = -x \sin 2x$$

BONUS
Solve

$$x^2 y'' + 4xy' + 2y = \ln x, \quad y(1) = 2, \, y'(1) = 0$$

SOLUTIONS

Problem 1
i) Solve first the homogeneous equation

$$y'' - 2y' + y = 0,$$

whose auxiliary equation is $\lambda^2 - 2\lambda + 1 = 0 \leftrightarrow (\lambda - 1)^2 = 0, \; \lambda_{1,2} = 1$. Thus $y_h = c_1 e^x + c_2 x e^x$. By variation of parameters, $y_p = u_1(x)e^x + u_2(x)xe^x$. We have the conditions:

$$u_1' e^x + u_2' x e^x = 0$$

$$u_1' e^x + u_2'(x+1)e^x = e^x \ln x$$

Solving for $u_1',\ u_2'$, we get

$$u_1 = -\int x \ln x dx = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x$$

$$u_2 = \int \ln x dx = x \ln x - x$$

With these, the particular solution becomes

$$y_p(x) = \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right) e^x + (x \ln x - x)\, xe^x = \frac{1}{2}x^2 e^x \ln x - \frac{3}{4}x^2 e^x$$

and the general solution is

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 xe^x + \frac{1}{2}x^2 e^x \ln x - \frac{3}{4}x^2 e^x.$$

Problem 2

Using reduction of order, we set $y_2(x) = u(x)x^{-1}$. Plug back into the equation $y_2' = u'x^{-1} - ux^{-2},\ y_2'' = u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$, we get

$$2xu'' - u' = 0$$

with solution$\ln u' = \frac{1}{2}\ln x$, giving $u' = x^{\frac{1}{2}},\ u = \frac{2}{3}x^{\frac{3}{2}}$ and thus $y_2 = x^{\frac{1}{2}}$, and thus

$$y(x) = c_1 x^{-1} + c_2\sqrt{x}$$

Problem 3

i) Solve the homogeneous equation first

$$y'' + 4y = 0$$

The auxiliary equation gives $\lambda^2 + 4 = 0,\quad \lambda_{1,2} = \pm 2i$. Then the solution to the homogeneous equation is $y_h = c_1 \cos 2x + c_2 \sin 2x$. Solving the inhomogeneous equation by the method of Undetermined Coefficients, set

$$y_p = x(Ax + B)\cos 2x + x(Cx + D)\sin 2x$$

(Note the multiplication by x since y_p is a repeated solution of y_h). We get:

$$y' = [2Cx^2 + 2(D + A)x + B]\cos 2x + [-2Ax^2 + 2(C - B)x + D]\sin 2x$$

and

$$y'' = [-4Ax^2+4(2C-B)x+4D+2A]\cos 2x+[-4Cx^2-4(D+2A)x-4B+2C]\sin 2x$$

and plugging into the equation $y'' + 4y = -x\sin 2x$, we identify the coefficients:

$$\begin{aligned}
x^2\cos 2x: &\quad -4A + 4A = 0\\
x^2\sin 2x: &\quad -4C + 4C = 0\\
x\cos 2x: &\quad 4(2C - B) + 4B = 0 \to C = 0\\
x\sin 2x: &\quad -4(D + 2A) + 4D = -1 \to A = \frac{1}{8}\\
\cos 2x: &\quad 4D + 2A = 0 \to D = -\frac{1}{16}\\
\sin 2x: &\quad -4B + 2C = 0 \to B = 0
\end{aligned}$$

Thus $y_p = \frac{1}{8}x^2\cos 2x - \frac{1}{16}x^2\sin 2x$, and

$$y(x) = (c_1 + \frac{1}{8}x^2)\cos 2x + (c_2 - \frac{1}{16}x^2)\sin 2x$$

BONUS

i) Find a solution to the homogeneous equation

$$x^2y'' + 4xy' + 2y = 0$$

which, using the change of variables $z = \ln x$, becomes

$$\frac{d^2y}{dz^2} + (4-1)\frac{dy}{dz} + 2y = 0$$

with auxiliary equation $\lambda^2 + 3\lambda + 2 = 0 \leftrightarrow (\lambda+1)(\lambda+2) = 0$. Thus $y_h = c_1e^{-z} + c_2e^{-2z} = c_1x^{-1} + c_2x^{-2}$.

ii) We can now use either undetermined coefficients for

$$\frac{d^2y}{dz^2} + (4-1)\frac{dy}{dz} + 2y = z,$$

or variation of parameters for the original equation in *standard form*

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{\ln x}{x^2}.$$

Choosing the first (for simplicity), $y_p = Az + B,\ \frac{dy}{dz} = A,\ \frac{d^2y}{dz^2} = 0$

$$3A + 2Az + 2B = z,$$

gives $A = \frac{1}{2},\ B = -\frac{3}{4}$. The general solution becomes

$$y(x) = c_1 e^{-z} + c_2 e^{-2z} = c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2}\ln x - \frac{3}{4}$$

Applying the initial conditions, $y(1) = c_1 + c_2 - \frac{3}{4} = 2$ and $y' = -c_1 - 2c_2 + \frac{1}{2} = 0$ gives $c_1 = 5$ and $c_2 = -\frac{9}{4}$, so the solution to the IVP is

$$y(x) = 5x^{-1} - \frac{9}{4}x^{-2} + \frac{1}{2}\ln x - \frac{3}{4}$$