

MATH 205 Final Exam December 2012

Solutions (Questions 1 to 5 only)

[10] 1. (a) Sketch a graph of the function

$$f(x) = \begin{cases} -\sqrt{4-x^2} & \text{for } |x| \leq 2 \\ |x-3|-1 & \text{for } 2 < x \end{cases}$$

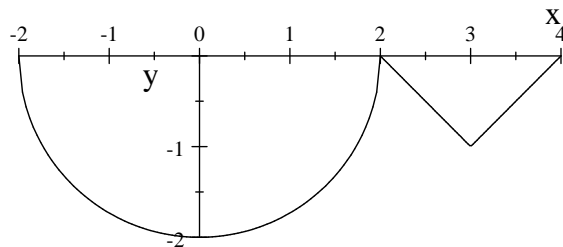
on the interval $-2 \leq x \leq 4$ and calculate the definite integral $\int_{-2}^4 f(x) dx$ in terms of signed area (*do not* antidifferentiate).

(b) Find the derivative $F'(x)$ of the function $F(x) = \int_{x^3}^1 \sqrt{1+t} \cos(\pi t) dt$, and use it to determine whether $F(x)$ is increasing or decreasing at $x = 1$.

[10] 2. Find the antiderivative $F(x)$ of the function $f(x)$ that satisfies the given condition:

(a) $f(x) = \frac{5^x}{5^x + 1}$, $F(0) = 1$. (b) $f(x) = \frac{\sec^2 x}{(1 + \tan x)^3}$, $F\left(\frac{\pi}{4}\right) = 0$.

Solution (a) The sketch:



Clearly the integral is the negative of the area $= -\frac{1}{2} (\pi \cdot 2^2 + 2 \cdot 1) = -2\pi - 1$ sq. units

(b) If $F(x) = \int_{x^3}^1 \sqrt{1+t} \cos(\pi t) dt = -\int_1^{x^3} \sqrt{1+t} \cos(\pi t) dt$ then $F'(x) = -\sqrt{1+x^3} \cos(\pi x^3) \cdot 3x^2$ by the Chain Rule. Since $F'(1) = -\sqrt{2} \cos(\pi) \cdot 3 = 3\sqrt{2} > 0$, the function F is increasing at $x = 1$.

Solution (a) $F(x) = \int \frac{5^x}{5^x + 1} dx$ with $F(0) = 1$. First find $I = \int \frac{5^x}{5^x + 1} dx$ by substitution: let $u = 5^x + 1$; $du = \ln 5 \cdot 5^x dx$ so $I = \frac{1}{\ln 5} \int \frac{du}{u} = \frac{1}{\ln 5} \ln(5^x + 1) + C$. To get $F(x)$ we put $x = 0$ and solve for C : $\frac{1}{\ln 5} \ln(5^0 + 1) + C = 1$ so $C = 1 - \frac{\ln 2}{\ln 5}$ and $F(x) = \frac{1}{\ln 5} \ln(5^x + 1) + 1 - \frac{\ln 2}{\ln 5}$.

(b) $F(x) = \int \frac{\sec^2 x}{(1 + \tan x)^3} dx$ with $F\left(\frac{\pi}{4}\right) = 0$. First find $I = \int \frac{\sec^2 x}{(1 + \tan x)^3} dx = \int \frac{du}{u^3}$ (with $u = 1 + \tan x$; $du = \sec^2 x dx$) $= -\frac{1}{2u^2} + C = -\frac{1}{2(1 + \tan x)^2} + C$. $F\left(\frac{\pi}{4}\right) = 0$ implies $-\frac{1}{2(1+1)^2} + C = 0$ so $C = \frac{1}{8}$.

3. Find the following indefinite integrals:

(a) $\int \frac{\ln x}{x^2} dx$ (b) $\int \frac{x}{x^2 - 2x - 3} dx$ (c) $\int (1 - e^x)^2 dx$

Solution (a) Integration by parts: $u = \ln x$ $dv = x^{-2} dx$
 $du = \frac{dx}{x}$ $v = -x^{-1}$ gives $\int \frac{\ln x}{x^2} dx = -\frac{1}{x} \ln x - \int \left(-\frac{1}{x}\right) \frac{dx}{x} = -\frac{1}{x} \ln x + \int \frac{1}{x^2} dx = -\frac{1}{x} \ln x - \frac{1}{x} + C$

(b) $\int \frac{x}{x^2 - 2x - 3} dx = \int \frac{x}{(x-3)(x+1)} dx$. Use partial fractions: $\frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$. Put $x = -1$ to get $-1 = -4B$ so $B = \frac{1}{4}$; Put $x = 3$ to get $3 = 4A$ so $A = \frac{3}{4}$ and the integral is

$$\frac{3}{4} \int \frac{dx}{x-3} + \int \frac{dx}{x+1} = \frac{3}{4} |\ln|x-3|| + \frac{1}{4} \ln|x+1| + C$$

(c) $\int (1 - e^x)^2 dx = \int (1 - 2e^x + e^{2x}) dx = x - 2e^x + \frac{1}{2}e^{2x} + C$

4. Find the following definite integrals:

(a) $\int_1^e \frac{1}{x(1 + \ln^2 x)} dx$ (b) $\int_0^{\pi/2} x^2 \cos(2x) dx$

Solution (a) First put $I = \int \frac{1}{x(1 + \ln^2 x)} dx$ (indefinite intrgral) and try substitution. Put $u = \ln x$; $du = \frac{dx}{x}$ and we get $I = \int \frac{1}{1 + u^2} du =$

$\arctan u + C$. Now we can go back to x or continue working with u . If we work with u we have to realize that when $x = 1$, $u = 0$ and when $x = e$, $u = 1$. So we want $\int_0^1 \frac{1}{1+u^2} du = \arctan u|_0^1 = \frac{\pi}{4}$.

- (b) Again, first put $I = \int x^2 \cos(2x) dx$ to find the antiderivative before plugging in the limits. This is clearly an integration by parts question

(twice) so
$$\begin{array}{l} u = x^2 \quad dv = \cos(2x) dx \\ du = 2x dx \quad v = \frac{1}{2} \sin(2x) \end{array} \quad \text{gives } I = \frac{1}{2} x^2 \sin(2x) -$$

$\int x \sin(2x) dx$. Again, put
$$\begin{array}{l} U = x \quad dV = \sin(2x) dx \\ dU = dx \quad V = -\frac{1}{2} \cos(2x) \end{array} \quad \text{to get } \int x^2 \cos(2x) dx$$

$$I = \frac{1}{2} x^2 \sin(2x) - \left(-\frac{1}{2} x \cos(2x) + \frac{1}{2} \int \cos(2x) dx \right) = \left(\frac{1}{2} x^2 - \frac{1}{4} \right) \sin(2x) +$$

$$\frac{1}{2} x \cos(2x). \text{ Now putting in the limits we get } \left(\left(\frac{1}{2} x^2 - \frac{1}{4} \right) \sin(2x) + \frac{1}{2} x \cos(2x) \right) \Big|_0^{\pi/2} =$$

$$\left(0 - \frac{\pi}{4} \right) - (0) = -\frac{\pi}{4}$$

The remaining question will be done in class
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