STAT 249/2 (2016)

Sample questions for final

1. Let A and B be two events. Suppose $P(A) = P(B|A) = 0.3$ and $P(B) = 0.2$. Are A and B independent? Find $P(\overline{A} \cap \overline{B})$.

   SOLUTION: Not independent since $P(B|A) \neq P(B)$.

   $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B|A)$.

2. A bowl contains 5 white balls and 4 black balls. One ball is selected at random from the bowl and labelled A, its color is noted, and it is returned to the bowl along with 3 additional balls of the same color. Now the bowl contains 12 balls. Another two balls are then randomly selected (without replacement) from the bowl and it is observed that one ball is white and the other ball is black. Find the conditional probability that the ball labelled A was white.

   SOLUTION: By Bayes' Theorem,

   P(A white | one white, one black at 2nd draw)
   $= P$(A white)P(one white, one black at 2nd draw | A white)/
   [P(A white)P(one white, one black at 2nd draw | A white)
   $+ P$(A black)P(one white, one black at 2nd draw | A black)]
   $= (5/9) \times (5+3) \times 4/\binom{12}{2}/$
   $\left[(5/9) \times (5+3) \times 4/\binom{12}{2} + (4/9) \times 5 \times (4+3)/\binom{12}{2}\right]$.

3. An airline knows that 95% of the customers with reservations for a certain flight will actually show up for the flight.

   (a) If the airline sold 52 reservations and the flight has only 50 seats, what is the chance that there will be at least one customer who cannot get a seat on the flight?

   SOLUTION: P(51 or more customers show up) $= \binom{52}{51}(0.95)^{51}(0.05) + \binom{52}{52}(0.95)^{52} = 52 \times (0.95)^{51}(0.05) + (0.95)^{52}$.

   (b) Suppose the airline allowed as many people as wanted to make the reservations, and made a list of people who made reservations. What is the chance that the first customer who does not show up for the flight has a rank on the airline list that is higher than 50?

   SOLUTION: P(first 50 customers show up) $= (0.95)^{50}$.

4. If you buy 5000 different lotteries and in each lottery your chance of winning a prize is 1/10000, you can approximate your number of winnings by a Poisson distribution.

   (a) What is the chance that you win at least one lottery?

This study resource was shared via CourseHero.com
https://www.coursehero.com/file/22301750/stat249-16-sampfinal/

SOLUTION: Approximate Binomial ($n = 5000, p = 0.0001$) by Poisson ($\lambda = np = 0.5$). Thus $P(\text{win at least one lottery}) = 1 - P(\text{win no lottery}) = 1 - e^{-0.5}$.

(b) If each lottery win is \$1000, what is the expected value and the variance of your winnings?

SOLUTION: Expectation $= \$1000 \times \lambda = \$500$, variance $= \$1000^2 \times \lambda = \$^2 500,000$.

5. Suppose X is the number you get after throwing a fair six sided die.

(a) Find the moment generating function of X.

SOLUTION:

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{6} e^{tx} P[X = x] = \sum_{x=1}^{6} e^{tx}/6 = \frac{e^t(e^{6t} - 1)}{6(e^t - 1)}.$$

(b) Use the moment generating function from (a) to calculate the expected value and the variance of X.

SOLUTION: $E(X) = (d/dt)M_X(t)|_{t=0} = (d/dt)[(e^t + \cdots + e^{6t})/6]_{t=0} = (1 + \cdots + 6)/6 = 7/2 = 3.5$, and $E(X^2) = (d^2/dt^2)M_X(t)|_{t=0} = (d^2/dt^2)[(e^t + \cdots + e^{6t})/6]_{t=0} = (1^2 + \cdots + 6^2)/6 = 91/6$, so $V(X) = (91/6) - (7^2/2^2)$.

6. Each of 3 balls are randomly placed into one of the 3 bowls A, B or C. Let Y = number of empty bowls.

(a) Find the probability distribution of Y.

SOLUTION: We have 3 choices (A, B, C) for placing each of the balls, hence the sample space is

$$\mathcal{S} = \{(A, A, A), (A, A, B), \ldots, (C, C, C)\}, \#(\mathcal{S}) = 3 \times 3 \times 3 = 27.$$

Then
$P(Y = 0) = P(\{(A, B, C), (A, C, B), \ldots, (C, B, A)\}) = 3!/27 = 6/27$;
$P(Y = 1)$
$= 3 \times P(\text{only } A \text{ is empty})$
$= 3 \times [P(B \text{ chosen 2 times, } C \text{ chosen once})$
$+ P(B \text{ chosen once, } C \text{ chosen 2 times})]/27$
$= 3 \times [\#(\{(B, B, C), (B, C, B), (C, B, B)\}) + \#(\{(C, C, B), (C, B, C), (B, C, C)\})]/27 = 18/27$;
$P(Y = 2) = \#(\{(A, A, A), (B, B, B), (C, C, C)\})/27 = 3/27$.

This study resource was shared via CourseHero.com
https://www.coursehero.com/file/22301750/stat249-16-sampfinal/

(b) Find the conditional probability $P(\text{B is empty}|Y \geq 1)$.

<u>Solution:</u> B is empty implies $Y \geq 1$, i.e., the event $\{\text{B is empty}\} \subset \{Y \geq 1\}$. Hence $P(\text{B is empty}|Y \geq 1) = P(\text{B is empty})/P(Y \geq 1) = P(\text{A or C chosen all 3 times})/P(Y \geq 1) = (2^3/27)/(21/27) = 8/21$.

7. Let Y be a continuous random variable with p.d.f.: $f(y) = 6y^2/5$ for $0 \leq y \leq 1, f(y) = 6(2-y)/5$ for $1 < y \leq 2$ and 0 elsewhere. Find
$P(|Y-1| \leq 0.5|Y \geq 1)$ and $E(2-Y)$.

<u>Solution:</u>

$P(|Y-1| \leq 0.5|Y \geq 1)$
$= P(|Y-1| \leq 0.5, Y \geq 1)/P(Y \geq 1)$
$= P(0.5 \leq Y \leq 1.5, Y \geq 1)/P(Y \geq 1)$
$= P(1 \leq Y \leq 1.5)/P(Y \geq 1)$
$= \int_1^{1.5}(6(2-y)/5)dy / \int_1^2 (6(2-y)/5)dy$
$= 3/4.$

$E(2-Y) = (6/5)[\int_0^1 (2-y)y^2 dy + \int_1^2 (2-y)^2 dy$

8. If a student's score on a math test has a Normal ($\mu = 18,\ \sigma^2 = 6^2$) distribution, determine a passing score s such that a student passes the test with probability 0.7517 and a number c such that a student's score lies between $18-c$ and $18+c$ with probability 0.99, using the Normal $(0, 1)$ table provided.

<u>Solution:</u> Straightforward.

9. A committee of three people is to be randomly selected from a group containing two sophomores, three juniors, and four seniors. Let X and Y denote numbers of juniors and seniors, respectively, on the committee.

(a) Find the joint probability distribution for X and Y.

(b) Calculate $P(X \leq 1|Y = 1)$.

<u>Solution:</u> (a)

$$P(X=0,Y=1) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{9}{3}},\quad P(X=0,Y=2) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{9}{3}},\quad P(X=0,Y=3) = \frac{\binom{4}{3}}{\binom{9}{3}},$$

$$P(X=1,Y=0) = \frac{\binom{2}{2}\binom{3}{1}}{\binom{9}{3}},\quad P(X=1,Y=1) = \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}},\quad P(X=1,Y=2) = \frac{\binom{3}{1}\binom{4}{2}}{\binom{9}{3}},$$

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/22301750/stat249-16-sampfinal/

$$P(X=2,Y=0) = \frac{\binom{2}{1}\binom{3}{2}}{\binom{9}{3}}, \quad P(X=2,Y=1) = \frac{\binom{3}{2}\binom{4}{1}}{\binom{9}{3}},$$

$$P(X=3,Y=0) = \frac{\binom{3}{3}}{\binom{9}{3}}.$$

(b)

$$P(X \le 1 | Y=1) = \frac{P(X \le 1, Y=1)}{P(Y=1)} = \frac{P(X=0,Y=1)+P(X=1,Y=1)}{P(X=0,Y=1)+P(X=1,Y=1)+P(X=2,Y=1)}.$$

Using part (a), we can find the value of $P(X \le 1 | Y=1)$.

10. Suppose (X,Y) has the joint density

$$f(x,y) = \begin{cases} 6\exp(-3x-2y), & x \ge 0 \text{ and } y \ge 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities of X and Y. Are X and Y independent?

(b) Calculate $P(X \le 2Y)$.

<u>SOLUTION:</u> (a)

$$f_X(x) = \int_0^\infty 6\exp(-3x-2y)dy = 3\exp(-3x), \quad x \ge 0,$$

$$f_Y(y) = \int_0^\infty 6\exp(-3x-2y)dx = 2\exp(-2y), \quad y \ge 0.$$

Since $f(x,y) = f_X(x)f_Y(y)$ for any x and y, X and Y are independent.

(b)

$$\begin{aligned} P(X \le 2Y) &= \int_0^\infty \int_0^{2y} 6\exp(-3x-2y)dxdy \\ &= \int_0^\infty 2\exp(-2y)\left(-\exp(-3x)|_0^{2y}\right)dy \\ &= \int_0^\infty 2\exp(-2y)(1-\exp(-6y))dy \\ &= \int_0^\infty [2\exp(-2y) - 2\exp(-8y)]dy \\ &= 1 - \frac{2}{8} \\ &= \frac{3}{4}. \end{aligned}$$

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/22301750/stat249-16-sampfinal/