MAT 1330 - Calculus for the Life Sciences I

Notes — By Eric Hua

Contents

Introduction

Main Contents:

- Derivatives: product and quotient rules, chain rule, derivative of exponential, logarithm and basic trigonometric functions, higher derivatives, curve sketching.
- Applications of the derivative to life sciences.
- Discrete dynamical systems: equilibrium points, stability, cobwebbing.
- Integrals: indefinite and definite integrals, fundamental theorem of calculus, antiderivatives, substitution, integration by parts.
- Applications of the integral to life sciences.

Prerequisite: One of MAT1339, Ontario 4U Calculus and Vectors (MCV4U) or an equivalent. The courses MAT1330, MAT1300, MAT1308, MAT1320 cannot be combined for credits.

Precalculus Review

1. Real numbers and intervals

Interval Notation	**Set Notation**
$[a, b]$	$\{x \in \mathbb{R} : a \le x \le b\}$
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$
$[a, b)$	$\{x \in \mathbb{R} : a \le x < b\}$
$(a, b]$	$\{x \in \mathbb{R} : a < x \le b\}$
$(a, +\infty)$	$\{x \in \mathbb{R} : x > a\}$
$[a, +\infty)$	$\{x \in \mathbb{R} : x \ge a\}$
$(-\infty, b)$	$\{x \in \mathbb{R} : x < b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} : x \le b\}$
$(-\infty, +\infty)$	$\mathbb{R}$

2. Solving inequalities

Example 1 *Solve the inequality*

$$-2x - 3 \le -13.$$

Example 2 *Solve the inequality*

$$x^2 + 2x - 35 < 0.$$

3. Absolute Values

Definition 1 *Let $x \in \mathbb{R}$. The* absolute value of x*—denoted by $|x|$—is defined by*

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \ge 0. \end{cases}$$

Example 3 *Let $x > 10$. Then $|3 - |5 - x|| = |3 - (x - 5)| = |8 - x|$.*

- $|x| \le t \iff -t \le x \le t.$
- $|x| \ge t \iff x \ge t \quad \text{or} \quad x \le -t.$
- Triangle Inequality: Let a, b be real numbers. Then $|a + b| \le |a| + |b|$.

Example 4 *Solve the inequality $|2x - 1| \le 1$.*

4. Exponents and radicals

Properties of exponents:

- $x^0 = 1, \quad x \neq 0.$
- $x^{-n} = \frac{1}{x^n}, \quad x \neq 0.$
- $x^{1/n} = \sqrt[n]{x}$, $x^{m/n} = \sqrt[n]{x^m}$.
- $x^m x^n = x^{m+n}$, $x^m/x^n = x^{m-n}$.
- $(x^m)^n = x^{mn}$.
- $x^n y^n = (xy)^n$.

For Example,

$$\frac{x^{3/2} + 5x^2}{x^{1/2}} = x(1 + 5x^{1/2}).$$

5. Factoring Polynomials

- $a^2 - b^2 = (a - b)(a + b)$.
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad \text{and} \qquad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
- $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2 y^{n-3} + xy^{n-2} + y^{n-1})$.
- $(a \pm b)^2 = a^2 \pm 2ab + b^2$.
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad \text{and} \qquad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Example 5

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 - x)(x^2 + 1 + x). \end{aligned}$$

Example 6 $x^2 - 8x - 9 = (x - 9)(x + 1)$.

6. Rationalizing denominator or numerator

- If the denominator is $\sqrt{a}$, then multiply both top and bottom by $\sqrt{a}$.
- If the denominator is $\sqrt{a} \pm \sqrt{b}$, then multiply both top and bottom by $\sqrt{a} \mp \sqrt{b}$.

Example 7

$$\frac{x}{\sqrt{x+4} - 2} = \frac{x(\sqrt{x+4} + 2)}{(\sqrt{x+4} - 2)(\sqrt{x+4} + 2)} = \frac{x(\sqrt{x+4} + 2)}{x} = \sqrt{x+4} + 2.$$

1.1–2.3 Functions and Models

Definition of a Function

Function: *A function $y = f(x)$ from a set D to a set R is a rule that assigns a unique element $f(x) \in R$ to each element $x \in D$. (x is called independent variable, y is called dependent variable).*

- Domain of the function $y = f(x)$: $D =$ The set of all values of the independent variable x for which the function is defined.

- Range of the function: $R =$ The set of all values taking on by the dependent variable y.

Example: $f(x) = \frac{x^2}{x^2-3x+2}$ is a function, $D = \{x : x \neq 1, 2\}$.
Example: $f(x) = \pm x^2$ is not a function.
Some special functions:

- Linear function: $y = f(x) = mx + b$.

- Increasing function $f(x)$: $f(x)$ increases as x increases.

- Decreasing function $f(x)$: $f(x)$ decreases as x increases.

- Piecewise defined functions: $f(x) = \begin{cases} 2x, & x \leq 0; \\ 3x, & x > 0. \end{cases}$

- Power function: $f(x) = kx^p$, where $k \neq 0$ and p are constants, e.g., $\sqrt{1 - x^2}$.

- Polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where n is a positive integer (which is called the degree of $P(x)$).

- Rational function: $f(x) = \frac{p(x)}{q(x)}$.

- Absolute value:

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Exponential functions

We say that $f(x) = a^x$ is an exponential function with base a.

- Domain: $x \in \mathbb{R}$; Range: $y > 0$.
- Exponential growth: $a > 1$; Exponential decay: $0 < a < 1$.
- Natural exponential function is defined as: $y = f(x) = e^x$, where $e \doteq 2.71828....$
- Exponential model: $f(x) = ce^{\alpha x}$, $c \neq 0$, $\alpha \neq 0$. Here α is the exponential growth/decay rate.
- Graph: e.g., $y = 2^x + 5$, $y = 2^{-x} + 5$.

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Example 8 *Solve for x:* $3^{2x-3} = 9^{1-3x}$, $2^{2x+1} - 9(2^x) + 4 = 0$.

Example 9 *The relationship between the length (inch) of Muskie fish and the weight (pound) can be modeled by*

$$W = 0.000089L^{3.325}.$$

E.g., $18lb \leftrightarrow 40in.$

Applications on population growth/decay: Let $P(t)$ be the population after t years.

- Half-life (exponential decay): The time required for the quantity to be reduced to half. Let H be the half-life, then

$$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$

- Doubling-time (exponential growth): The time required for the quantity to be doubled. Let D be the doubling time, then

$$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$

Example 10 *A bacterial culture starts with 500 bacteria and doubles in size every hour.*

a) How many are there after t hours?

b) How many are there after 10 minutes?

Logarithms

Inverse function: One-to-one function: $y = f(x)$ is 1-1 $\Leftrightarrow$ for each $y \in R$, there is only one $x \in D$. Horizontal line test can be used to check this.

Example 11 $f(x) = x^2$ *is not 1-1;* $g(x) = x^2$, $x > 0$ *is 1-1.*

Inverse function: $y = f(x) \to x = f^{-1}(y)$. We write it as $y = f^{-1}(x)$.

- The graph of f^{-1} and the graph of f are symmetric about the line $y = x$.
- Cancelation: $f(f^{-1}(y)) = y$.
- $f^{-1}(f(x)) = x$
- $D(f) = R(f^{-1})$, $R(f) = D(f^{-1})$.

Example 12 *let* $f(x) = \frac{3x+2}{5x-4}$, *find the inverse* $f^{-1}(x)$.

Strategy:
1) Write $y = \frac{3x+2}{5x-4}$;
2) Switch x and y: $x = \frac{3y+2}{5y-4}$;
3) Isolate y: $y = \frac{4x+2}{5x-3}$;
4) Answer: $y = f^{-1}(x) = \frac{4x+2}{5x-3}$.

$$y = a^x \quad \xrightarrow{inverse function} \quad y = \log_a x,$$
$$y = e^x \quad \xrightarrow{inverse function} \quad y = \log_e x = \ln x,$$
$$y = 10^x \quad \xrightarrow{inverse function} \quad y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ is called logarithmic function with the base a. Domain $= \{x > 0\}$.

Laws: Let $B, C > 0$. Then

1. $\log_a(BC) = \log_a B + \log_a C$,
2. $\log_a(\frac{B}{C}) = \log_a B - \log_a C$,
3. $\log_a(B^n) = n \log_a B$,
4. $\log_a(a^x) = x$, $\log_a a = 1$,
5. $a^{\log_a B} = B$,
6. $\log_a 1 = 0$.
7. Change of base: $\log_a b = \frac{\log_c b}{\log_c a}$.

Proof. Let $x = \log_a b$. Then $a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x \log_c a = \log_c b$.

Example 13 *Convert a^x to base e.*

$$a^x = e^{x \ln a}.$$

Example 14 *Simplify* $\log_3 18 - \log_3 2$.

Example 15 *Solve for x:*

$$3^{2x-1} = 4, \quad \ln[\ln(2x+1)] = 1, \quad \log_3 x + \log_3(x-8) = 2.$$

Example 16 *Sketch* $y = \ln(x+1) - 2$.

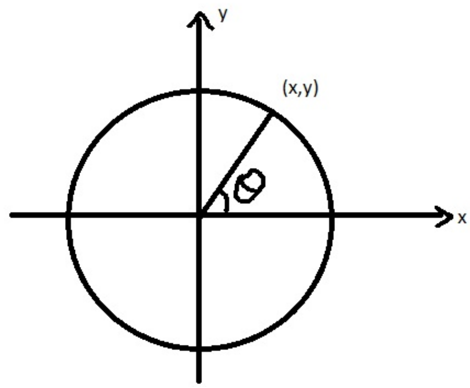
Example 17 *Predict the population in 2010, if*

Year	*Population*
2000	*10*
2003	*10.5*

Trigonometric functions

Radian $\leftrightarrows$ Degree: t degree $= \frac{t}{180}\pi$.

For any point (x, y), let $r = \sqrt{x^2 + y^2}$.



$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta},$$
$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

Pythagorean trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

Special values:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Double-angle formulas:

$$\sin 2x = 2\sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Half-angle formula.

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

Periods: $\sin x$ and $\cos x$ have period 2π, $\tan x$ and $\cot x$ have period π.

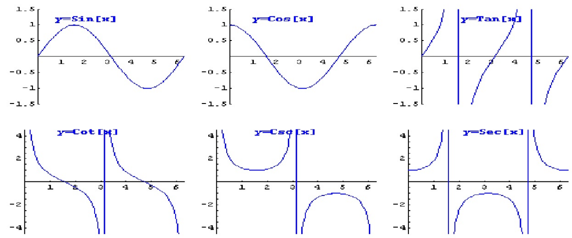
Example 18 *Sinusoidal function* $f(x) = 2\sin[3(x - \frac{\pi}{6})] + 1$.

Example 19 *Find all values of* x *in the interval* $[0, 2\pi]$ *such that* $\sin^2 x - 3\cos^2 x = 0$.

Example 20 *Find* $\cos x$ *where* $x \in [\frac{\pi}{2}, 2\pi]$ *such that* $\sin x = 0.8$.

Solution: $\cos x = -0.6$

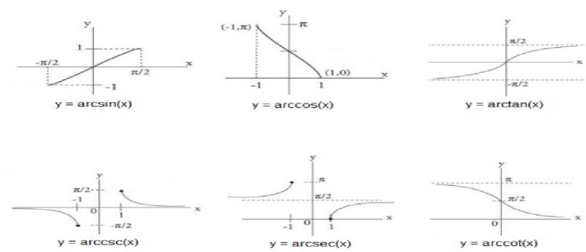
Graphs.



Inverse Trig Functions

Inverse Trig Function	Domain	Restriction (Range)	Meaning
$y = \arcsin x = \sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$\sin y = x$
$y = \arccos x = \cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$	$\cos y = x$
$y = \arctan x = \tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\tan y = x$
$y = arcsec\ x = \sec^{-1}(x)$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	$\sec y = x$
$y = arccsc\ x = \csc^{-1}(x)$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0$	$\csc y = x$
$y = arccot\ x = \cot^{-1}(x)$	$-\infty < x < \infty$	$0 < y < \pi$	$\tan y = x$

Graphs of the inverse functions: Using the symmetry line $y = x$ to get the graph for inverse from original functions.



Example 21 *Find the exact values of the following expressions: (a)* $\arcsin(1)$ *(b)* $\arctan(-1)$ *(c)* $\tan^{-1}(\sqrt{3})$ *(d)*$\sin[\cos^{-1}(\frac{\sqrt{3}}{2})]$ *(e)* $\arctan(\tan x)$*, where* $\frac{3\pi}{4} \leq x \leq 2\pi$*.*

Example 22 *Simplify the following expression:* $\tan \arcsin \frac{x}{a}$.

3.1–3.5 Discrete-Time Dynamical System (DTDS)

Introduction

The dynamic of any situation refers to how the situation changes over the course of time. A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next. One basic goal of the mathematical theory of dynamical systems is to determine or characterize the long-term behavior of the system. Often a physical setting is reduced to a set of measurements, for example, temperature, pressure, stock market prices, etc. In discrete-systems, we give these measurements at a sequence of specific times. We would hope that given the measurements at time t that we have a rule to determine the measurements at time $t+1$. If m_t represents the measurements at time t, this rule may take the form

$$m_{t+1} = f(m_t), \quad f^{-1}(m_{t+1}) = m_t,$$

where $f(x)$ is a given function fixed for all time, and is called **updating function**. This is referred as **recursion or recursive relation**. The **inverse** f^{-1} go one step into the past, which corrsponds to an "updating" function that goes backward in time.

Composition: $f \circ f$ = jump two time units into the future; $f \circ f \circ f$ = jump three time units into the future, ...

Solution and graph: The sequence m_0, m_1,... is the solution of the dynamical system. Graph = $\{m_t : t = 0, 1, 2, ...\}$.

Example 23 *Let $f(x) = 2x(1-x)$. The graph of this function is a parabola passing through the x-axis at $x = 0, 1$. The maximum value is 0.25 occurring at $x = .5$.*

We have some discrete systems like:

$$x_0 = 0, x_1 = 0 = ... = x_n = ...;$$

$$x_0 = 1, x_1 = 0 = ... = x_n = ...;$$

$$x_0 = 0.5, x_1 = 0.5 = ... = x_n = ...$$

$$f: \quad [0,1] \longrightarrow [0,1].$$

In general,

$$x_n = \underbrace{f \circ f \circ ... \circ f}_{n}(x_0).$$

Since $f(0) = 0$, $f(0.5) = 0.5$, so $x = 0$ and $x = 0.5$ are called fixed points of $f(x)$.

x_0	0.1
x_1	0.18
x_2	0.2952
x_3	0.41611392
x_4	0.4859262512
x_5	0.4996038592
x_6	0.4999996862
x_7	0.5000000000

We may easily guess the long-term behavior of this system:

$$\lim_{n \to \infty} x_n = 0.5.$$

Example 24 *Let* $x_{t+1} = 3x_t^2$, $x_0 = 0.2$. *Find* $f(x)$ *and* x_{100}.

Example 25

- *Basic exponential discrete-time dynamical system:* $b_{t+1} = rb_t$, $b_t = b_0 r^t$.
- *Basic additive discrete-time dynamical system:* $h_{t+1} = a + h_t$, $h_t = h_0 + at$.

Example 26 *Dynamics of absorption of pain medication: Let* M_t *be the amount of methadone in the patient's body at time* t. *Due to absorption,* M_t *is reduced to half within a day. Administering a new dosage will increase that amount by 1. Then the model is*

$$M_{t+1} = 0.5M_t + 1.$$

Analysis of DTDS

Cobwebbing: A graphical solution technique

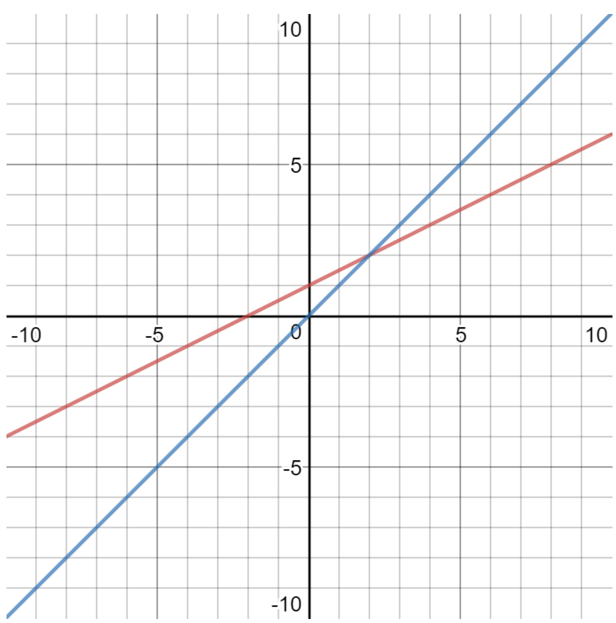
Given the discrete-time dynamical system

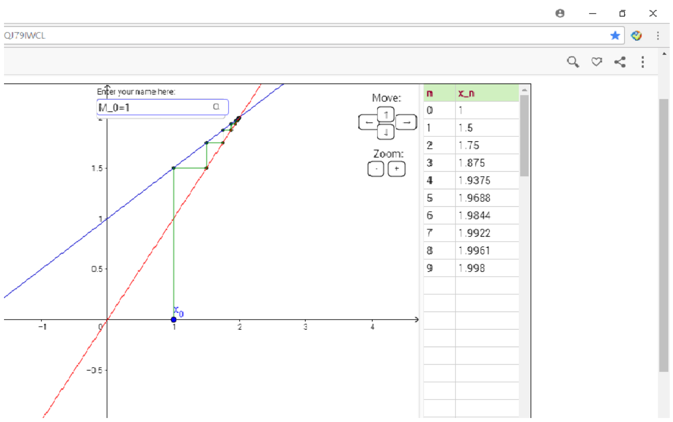
$$x_{t+1} = f(x_t)$$

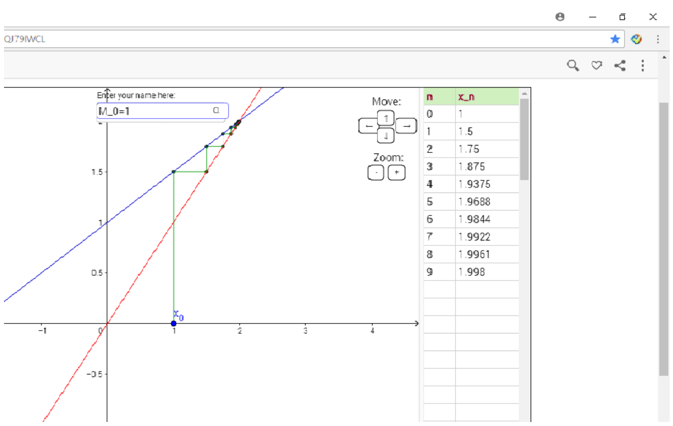
and **initial condition** x_0, we want to find other points on the curve $y = f(x)$.
Strategy:

1. Draw the updating function and the diagonal line $y = x$;
2. x_1 is the coordinate of the vertical point on the graph directly above x_0, so we get (x_0, x_1).
3. Move the point (x_0, x_1) horizontally until it intersects the diagonal line, we get the intersection (x_1, x_1).
4. Move the intersection vertically until it intersects the graph, we get x_2, then repeat...
5. **Sketch the solutions at times 0, 1, 2, ...**

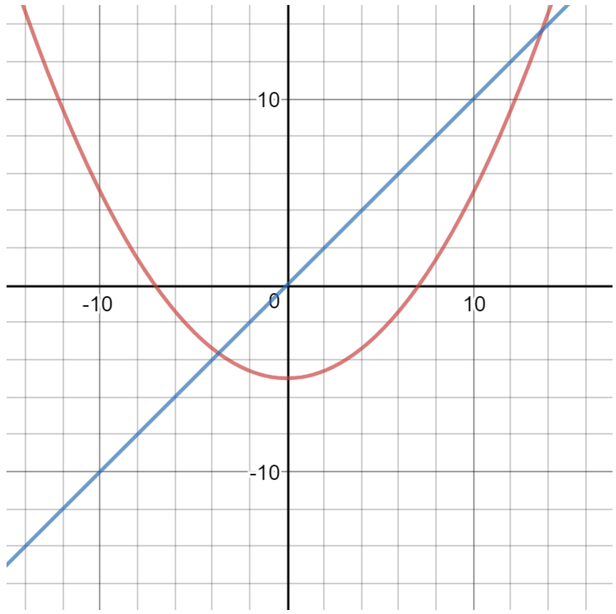
Example 27 *Cobweb the pain medication model with* $M_0 = 1$ *and* $M_0 = 4$.







Example 28 *Cobweb* $x_{t+1} = 0.1x_t^2 - 5$ *with* $x_0 = 5$.



t	0	1	2	3
x_t	5	-2.5	-4.375	-3.0859

Equilibrium (or fixed point):

Definition 2 *A point m^* is called an equilibrium (or a fixed point) of the discrete-time dynamical system*

$$x_{t+1} = f(x_t)$$

if

$$f(m^*) = m^*.$$

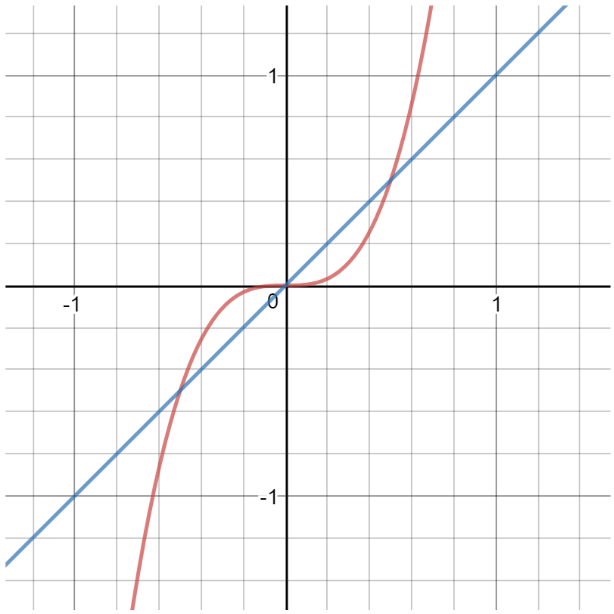
Remark. At any equilibrium, $f(x)$ neither increases nor decreases, remains the same. The above definition gives you **Algebraic Approach** to find equilibria.

Graphic approach: The intersections of the updating function and the line $y = x$.

Example 29 $x_{t+1} = 4x_t^3$. *Find all equilibria.*

Stability of equilibrium: An equilibrium m^* is called stable if the solutions that start near m^* stay near or approach m^*; if the solutions that start near m^* moves away from it, then m^* is unstable.

Example 30 $x_{t+1} = 4x_t^3$. *Study the stabilities of all equilibria.*



t	0	1	2	3
x_t	0.2	0.032	0.0001	0

t	0	1	2	3
x_t	0.6	0.864	2.5799	68.6853

t	0	1	2	3
x_t	-0.6	-0.864	-2.5799	-68.6853

Modeling with DTDS

Absorption of Caffeine: By c_t we denote the amount (in mg) of caffeine at time t (in hours). On average, our body eliminates 13% per hour. Assume that at the end of the same time interval we consume d extra mg of caffeine, then the model will be:

$$c_{t+1} = 0.87c_t + d.$$

Example 31 *Find the half life with $d = 0$.*

Example 32 *Find the general solution of the DTDS $c_{t+1} = ac_0 + d$.*

$$c_t = \left(c_0 - \frac{d}{1-a}\right)(a^t) + \frac{d}{1-a}.$$

Population growth/decay: By b_t we denote the amount of bacterial at time t. Consider the model:

$$b_{t+1} = rb_t,$$

where r represents the number of new bacterial produced per bacterium, called the **per capita production**.

Example 33 *If the population doubles each hour, then $r = 2$; if the population decreases by 50% each hour, then $r = 1/2$.*

Alcohol Use: We define a unit of alcohol as: **one drink** contains 14 g of alcohol, which is equivalent to 44 mL of rum, or 144 mL of white wine, or 355 mL of beer. Let a_t be the amount of alcohol (in grams) at time t, let $r(a_t)$ be the rate of elimination when the amount of alcohol in the body is a_t. Then

$$r(a_t) = \frac{10.1}{4.2 + a_t}, \quad a_t \geq 5.9g.$$

Then

$$a_{t+1} = a_t - a_t r(a_t) + d(new\ amount) = a_t - \frac{10.1a_t}{4.2 + a_t} + d.$$

Example 34 *Assume that someone has two rapid drinks and then decides to consume half a drink every hour. What will the long-term effects be?*

Nonlinear Dynamics Model of Selection

Discrete-time dynamical system is

- **liner**, if the updating function is linear;
- **nonliner**, if the updating function is nonlinear.

A model of selection: Let b_t and m_t be the population of bacterial and mutant respectively, at time t. Assume that

- **bacterial:** $b_{t+1} = rb_t$;
- **mutants:** $m_{t+1} = sm_t$.

If $s > r$, over time, the population of mutants will be larger and larger. The establishment of this mutant is an example of **selection**.

Modeling the dynamics of the fraction: Let p_t be the fraction of mutants at time t. Then

$$p_t = \frac{m_t}{m_t + b_t},$$

$$p_{t+1} = \frac{m_{t+1}}{m_{t+1} + b_{t+1}} = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

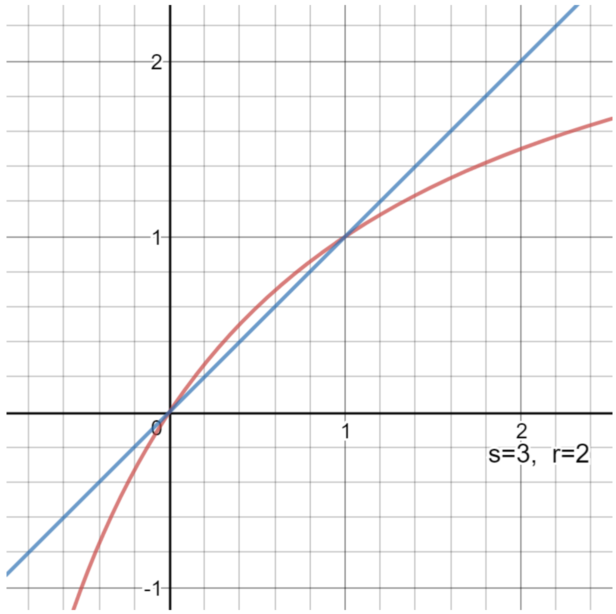
The updating function is

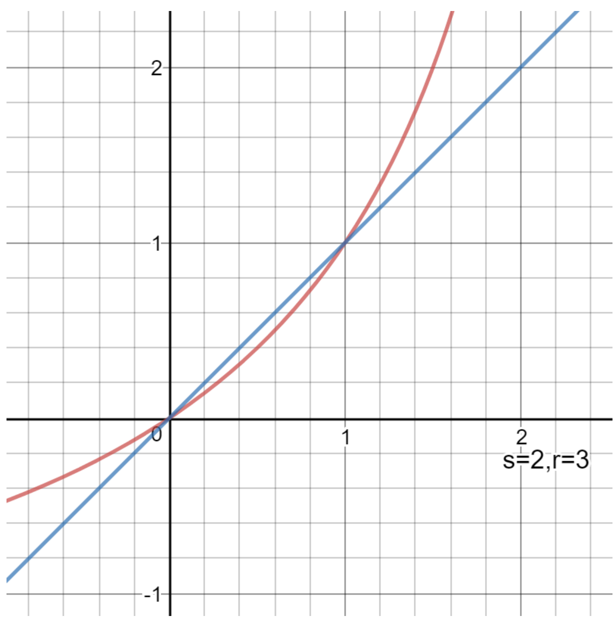
$$f(p_t) = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

Equilibria are $p^* = 0, 1$(when $s \neq r$).

Stability of the equilibria:

- $p^* = 0$ **is unstable:** if $p_0 = 0.1$, then $(t, p_t) = (0, 0.1), ..., (\infty, 1)$;
- $p^* = 1$ **is stable:** if $p_0 = 0.8$, then $(t, p_t) = (0, 0.8), ..., (\infty, 1)$.


s=3, r=2


s=2,r=3

4.1–4.3 Limits

The Tangent and Velocity Problem

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0.$$

Geometrically, it is the slope of the secant through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Instantaneous rates of change and tangent lines: What is a tangent line at point P on a curve? We chose another point Q on the curve. The line PQ is called a secant line. When Q tends to P, the secant PQ will tends to a line, which is called a the tangent line of the curve at P.

Definition 3 *Let $s = f(t)$ be position function.*

$$\text{average velocity} = \frac{\text{total distance}}{\text{total time}} = \frac{\Delta s}{\Delta t}.$$

Example 35 *Consider the position function $s = t^2 - 3t + 5$. Find the average velocity from $t = 3$ to $t = 4$.*

The Limit of A Function

Definition 4 *We write*

$$f(a - 0) = \lim_{x \to a^-} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the left. Similarly, We write

$$f(a + 0) = \lim_{x \to a^+} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the right.

Example 36 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2^-} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0.$$

Example 37 $\lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$

Example 38 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x \to 0^-} f(x) = -5$ *and* $\lim_{x \to 1^+} f(x) = 4.$

Definition 5 *We write*

$$\lim_{x \to a} f(x) = L$$

and say "as x *approaches* a*, the limit of* $f(x)$ *is* L*." If* L *is a finite number, we say that the limit exists, otherwise, the limit does not exist.*

Theorem 1

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Example 39 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0, \Rightarrow \lim_{t \to 0} H(t) \nexists.$$

Example 40 $\lim_{x \to 0} \frac{|x|}{x} \nexists.$

$$\because \lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Example 41 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \le x \le 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x\to 0} f(x) \nexists$ *and* $\lim_{x\to 1} f(x) = 4$.

Euler's Number e

$$e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}(1+\frac{1}{x})^x = 2.71828....$$

Example 42 *Calculate*

$$\lim_{x\to 0}(1-x)^{1/x}, \quad \lim_{x\to\infty}(1+\frac{3}{x})^x.$$

Example 43 *Evaluate*

$$\lim_{x\to 0^-} e^{1/x}.$$

Limit Laws: Suppose that $\lim_{x\to a} f(x)\ \exists$ and $\lim_{x\to a} g(x)\ \exists$.

- $\lim_{x\to a} P(x) = P(a), \quad P(x)$ is a polynomial.
- $\lim_{x\to a}(cf(x) \pm dg(x)) = c\lim_{x\to a} f(x) \pm d\lim_{x\to a} g(x),$ c, d are constants.
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad if \ \lim_{x\to a} g(x) \neq 0.$
- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n.$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$. When n is even, we assume that $\lim_{x\to a} f(x) \neq 0$.

Example 44

$$\lim_{x\to 1}(x^2-3) = 1^2 - 3 = -2, \quad \lim_{x\to 1}\frac{3x^4+8x-2}{x-2} = \frac{3(1)^4+8(1)-2}{1-2} = -9.$$

Special case:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \quad where\ g(a) = 0.$$

- If $f(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.
- If $f(a) = 0$, then simplify $\frac{f(x)}{g(x)}$ first, then study the limit.

Example 45

$$\lim_{x\to 2} \frac{3x^4 + 8x - 2}{x - 2} \nexists, \quad \lim_{x\to 2} \frac{x-2}{x-2} = 1.$$

Example 46

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2 - 4}{3 - |x - 5|} &= \lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2}(x + 2) = 4, \\
\lim_{h\to 0} \frac{(h+1)^2 - 1}{h} &= \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(h + 2) = 2, \\
\lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x} &= \lim_{x\to 0} \frac{(\sqrt{x+4} - 2)(\sqrt{x+4} + 2)}{x(\sqrt{x+4} + 2)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+4} + 2)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+4} + 2} = \frac{1}{4}.
\end{aligned}$$

Theorem 2 *If $f(x) \leq g(x)$ near $x = a$, then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 3 *The Sandwich Theorem (The Squeeze Theorem): If $f(x) \leq g(x) \leq h(x)$ near $x = a$, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Example 47 *Show that*

$$\lim_{x\to 0} x^4 \cos \frac{3}{x} = 0$$

by the Squeeze Theorem.

Example 48

$$\lim_{x\to 0} \sin x = 0, \quad \lim_{x\to 0} \cos x = 1.$$

Example 49 *Estimate the limit of*

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

x	$\frac{\sin x}{x}$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983

Famous result:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Proof. It is from the inequality

$$\cos x < \frac{\sin x}{x} < 1.$$

This will imply that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{\cos h + 1} = 0.$$

Example 50

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2x}{3x} = \frac{2}{3}.$$

Infinite Limits: Vertical Asymptote

Definition 6

$$\lim_{x\to a} f(x) = \infty$$

means that $f(x)$ can be arbitrarily large as x tends to a;

$$\lim_{x\to a} f(x) = -\infty$$

means that $f(x)$ can be arbitrarily large negative as x tends to a.

Example 51 $\lim_{x\to 0} \frac{1}{x^2} = \infty$, $\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$.

Definition 7 *The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x) = \pm\infty, \lim_{x\to a^+} f(x) = \pm\infty, \lim_{x\to a} f(x) = \pm\infty.$$

Example 52 *Find VA: $f(x) = \tan x$, $\ln x$.*

Limits at Infinity, HA

Definition 8 *The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

Example 53 $f(x) = \frac{3x^2-x-1}{2x^2+3x}$ *has horizontal asymptote* $y = \frac{3}{2}$.

Example 54 $\lim\limits_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \begin{cases} 0, & \text{if } n < m; \\ \frac{a_n}{b_n}, & \text{if } n = m; \\ \pm\infty, & \text{if } n > m. \end{cases}$.

Example 55 $\lim\limits_{x\to\infty} \sin x$, $\lim\limits_{x\to\infty} \cos x$ *do not exist.*

Example 56 *Find the horizontal asymptotes of the function* $f(x) = e^x$.

Sol: $\lim_{x\to-\infty} e^x = 0$. Thus, HA: $y = 0$.

Example 57 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+1} - x.$$

Example 58 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+5x+1} - x.$$

Example 59 $y = \tan^{-1} x$ *has horizontal asymptotes* $y = \frac{\pi}{2}$ *and* $y = -\frac{\pi}{2}$.

Infinite limits at ∞

The notation $\lim\limits_{x\to\infty} f(x) = \infty$ is used to indicate that the values of f(x) become large as x becomes large. Similar meanings are for

$$\lim_{x\to\infty} f(x) = -\infty, \lim_{x\to-\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty.$$

Example 60 $\lim\limits_{x\to\infty} x^5 = \infty$, $\lim\limits_{x\to-\infty} x^5 = -\infty$, $\lim\limits_{x\to\pm\infty} (x^3 - x^5) = \mp\infty$.

Example 61 $\lim_{x\to\infty} e^x = \infty$.

4.4 Continuity

Definition 9 *If $\lim_{x\to a} f(x) = f(a)$, then $f(x)$ is continuous at $x = a$, otherwise, $f(x)$ is discontinuous at $x = a$. If $f(x)$ is continuous at any point on an interval, then $f(x)$ is continuous on the interval. For the end points, we only need sided limits.*

Example 62 *Explore discontinuity from graph.*

Example 63 *Consider $f(x) = \frac{x^2-2x+1}{x-1}$ at $x = 1$. $f(x)$ is undefined at $x = 1$. But $\lim_{x\to 1} f(x) = 0$. So the discontinuous point $x = 1$ is* **removable** *if we define $f(1) = 0$.*

Example 64 *Determine the continuity of $f(x) = \frac{|x|}{x}$.*

Definition 10 *If $\lim_{x\to a^-} f(x) = f(a)$, then $f(x)$ is continuous from the left at $x = a$; if $\lim_{x\to a^+} f(x) = f(a)$, then $f(x)$ is continuous from the right at $x = a$.*

Example 65 *Determine the left and right continuity at $x = 0$:*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}.$$

Theorem 4 *If $f(x)$ and $g(x)$ are continuous at a, then*

$$f \pm g, fg, cf \text{ (c is a constant)}, \frac{f}{g} \text{ (if } g(a) \neq 0)$$

are continuous.

Theorem 5 *Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions and logarithmic functions are continuous in their domain.*

Theorem 6 *If $\lim_{x\to a} g(x) = b$ and $f(x)$ is continuous at b, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Furthermore, if $g(x)$ is continuous at a, and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Example 66 *Find k such that $f(x) = \begin{cases} x^3 + kx^2 - 5x, & x > 2; \\ \frac{x}{x-3}, & x \leq 2 \end{cases}$ is continuous at $x = 2$.*

Example 67 *The greatest integer function $[x]$.*

Example 68

$$\lim_{x\to 1} \arcsin(\frac{1-\sqrt{x}}{1-x}) = \arcsin(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}) = \arcsin(\frac{1}{2}) = \frac{\pi}{6}.$$

4.5 Derivatives

Definition 11 *The derivative of the function* $y = f(x)$ *is the function* $f'(x)$*:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Meaning of $f'(x)$:

- instantaneous rate of change of $f(x)$ at x, or
- rate of change of $f(x)$ at x, or
- the slope of the tangent line to the curve at x.

Example 69 *Find the slope and the equation of the tangent line to the curve*

$$y = f(x) = 3x^2 - 6x + 1$$

at the point $(2, 1)$*. Sketch the curve.*

Example 70 *Let* $f(x) = \sqrt{x-3}$*. Find* $f'(x)$ *and state the domains of* f *and* f'*.*

Example 71 *The volume of a sphere of radius* r *is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ *by definition. What's the meaning of this derivative?*

Example 72 *A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 2cm.*

Definition 12 *The function* f *is differentiable at* a *if* $f'(a)$ *exists. It is differentiable on an interval if* $f'(a)$ *exists for any* a *on the interval.*

Theorem 7 *If a function is differentiable at* $x = c$*, then the function is continuous at* $x = c$*.*

Example 73 $f(x) = |x|$ *is not differentiable at* $x = 0$*.*

Definition 13 *Critical number: A point p in the domain such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number.*

Example 74 *Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.*

What Does f' Say About f?

Definition 14 *$y = f(x)$ is increasing on an interval I if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$; $y = f(x)$ is decreasing on an interval I if $f(x_1) \geq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$.*

INCREASING/DECREASING TEST (I/D TEST):

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval, then f is a constant on that interval.

Example 75 *Let $f(x) = x^4 - 4x^3 + 4x^2 + 4$. State all the intervals of increase and decrease.*

Definition 15 *Let $s = f(t)$ be position function.*

$$\textit{average velocity} = \frac{\textit{total distance}}{\textit{total time}} = \frac{\Delta s}{\Delta t}.$$

Instantaneous velocity, or velocity, or rate of change at $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 76 *Consider the position function*

$$s = t^2 - 3t + 5.$$

Find the velocity at $t = 1$ and $t = 4$, interpret your results.

5.1–5.6 Differentiation Rules

Derivatives of Polynomials and Exponential Functions

- Constant rule: If $f(x) = c$, then $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.
- Power Rule: If $f(x) = x^n$, n is any real number. Then $f'(x) = nx^{n-1}$.
- Constant multiple rule: $[cf(x)]' = cf'(x)$.
- Sum rule and difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Derivative of polynomial: $[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0]' = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + a_1$.
- Derivative of exponential function:

$$(e^x)' = e^x.$$

Example 77 *Let $f(x) = a^x$, $a > 0$. Then*

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x f'(0).$$

Example 78 *Let $f(x) = 4x^3 + 6x^2 - 23x + 7$. Find the equation of the tangent line at $(1, -6)$.*

Example 79 *At what point(s) on the curve $y = e^x - x$ is the tangent line*

a) parallel to $y = 3x - 2$?

Solution:

b) perpendicular to $y = -\frac{1}{2}x$?

The product and quotient rules

- Product rule:
$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

- Quotient rule:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example. Let $f(x) = \sqrt{x}e^x$. Calculate $f'(4)$.
Example. Let $f(x) = \frac{\sqrt{x}+x^2}{e^x+x}$. Calculate $f'(4)$.
Example. Let $f(x) = \frac{x^3+4x^2}{x^5+x+1}$. Calculate $f'(1)$.

Example 80 *Let* $f(x) = \frac{x}{e^x}$. *Calculate* $f^{(n)}(x)$.

Example 81 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*
a) parallel to $y = 3x$*?*
b) perpendicular to $y = -0.5x$*?*

The chain rule

- Chain Rule:
$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv} \cdot \frac{dg(x)}{dx}, v = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- General Power Rule:
$$[u(x)^n]' = nu^{n-1}u'(x).$$

Example 82 *Let* $f(x) = (x^2 - x - 1)^{100}$. *Calculate* $f'(x)$.

Example 83 *Let* $h(x) = g(f(x))$, *where* $f'(2) = 3$, $f(2) = 4$, $g'(3) = -5$, $g(4) = 8$, $g'(4) = 7$. *Find* $h'(2)$.

Example 84 *Let* $y = \sqrt{x + \sqrt{x^2 + x}}$. *Calculate* y'.

Example 85 *Find* $f'(x)$*. If*

$$f(x) = \sin x^2, \quad \sin^2 x, \quad e^{\sin x}, \quad \sin(\cos(\tan x))).$$

Derivative of exponential functions:

$$(a^x)' = a^x \ln a.$$

Proof.

$$(a^x)' = (e^{\ln a^x})' = (e^{x \ln a})' = (e^{x \ln a})(x \ln a)' = a^x \ln a.$$

Derivative of Logarithmic Function

By using the formula

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))},$$

We can get some special results:

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}, \cdots$$

  Change base:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Example 86 *Differentiate* $\ln(x^2 + 1)$*.*

Logarithmic differentiation

Example 87 *Differentiate* $y = \frac{(x^2+x+5)\arcsin x}{(x+1)^2}$*.*

Example 88 *Differentiate* x^x*,* $(\sin x)^x$*.*

Number e

$$e = \lim_{x \to 0}(1 + x)^{1/x} = \lim_{x \to \infty}(1 + \frac{1}{x})^x.$$

Derivatives of Trigonometric Functions

Famous result:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

This will imply that

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0}\frac{\sin h}{h}\frac{\sin h}{\cos h + 1} = 0.$$

Derivative of Trig Functions:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x,$$

$$(\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x, \quad (\cot x)' = -\csc^2 x.$$

Example 89 *Differentiate* $\csc x$, $\cot x$, $e^x \cos(x)$, $\frac{1+\cos x}{1+\sin x}$, $e^x \sin x$.

Example 90 *Let* $y = \sin(x)$, *calculate* $y^{(10)}(x)$.

Example 91 *Given the position function* $s = f(t) = 2\sin(t)$, *calculate the velocity and acceleration at* $t = \frac{\pi}{3}$.

Example 92 *Find the equation of the tangent line to the curve* $\sin(1-x)$ *at* $(1, 0)$.

Hint: use the definition of the derivative of the function.

Implicit differentiation

Implicit Differentiation: Assume $f(x, y) = C$. To find y',

- consider x as an independent variable, y as a dependent variable;
- differentiate both sides with respect to x;
- isolate y'.

Example 93 *Find y' from $y^2 + x^2 = 1$.*

Example 94 *Let*

$$y^2 + x^2 = xy + 3.$$

1) Find the equation of the tangent line to the curve at $(0, \sqrt{3})$.
2) Find all the points on the curve where the tangent line is either horizontal or vertical.

Example 95 *Find y' from $\tan(xy + x) = x + y$.*

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arccos x = \frac{1}{-\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2},$$

$$\frac{d}{dx}\text{arcsec}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccsc}x = -\frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccot}x = -\frac{1}{1+x^2}.$$

Example 96 $y = \sin(\arctan 2x)$, $y = \arcsin\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

The Second Derivative, Concavity

Higher derivatives:

- Let $y = f(x)$. Then

$$y''(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx}\left(\frac{dy^{(n-1)}}{dx}\right).$$

- If $s(t)$ is a position function, then the velocity is $v(t) = s'(t)$, acceleration is $a(t) = v'(t) = s''(t)$.

Example 97 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Then* $f''(x) = 24x + 12$, $f'''(x) = 24$ *and* $f^{(4)}(x) = 0$.

Example 98 *Let* $f(x) = (x^2 - x - 1)^{100}$. *Calculate* $f''(x)$.

Example 99 *The position of a particle is given by*

$$s = t^3 - 15t^2 + 63t, \quad t \geq 0$$

where s is measured in meters and t in seconds.

a) What is the initial position? initial velocity? initial acceleration?

b) Find the velocity after 1s and 4s.

c) When is the particle at rest?

d) When is the particle moving in the positive direction?

e) When is the acceleration 0?

f) Find the displacement and the velocity at that time from e).

Definition 16 *(CONCAVITY)If the graph of* f *lies above all of its tangents on an interval I (*f' *is increasing on I), it is called concave upward on I. If the graph of* f *lies below all of its tangents on I (*f' *is decreasing on I), it is called concave downward on I. If* $f(x)$ *changes concavity at p, then p is an inflection point, and* $f''(p) = 0$ *or undefined.*

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Second Derivative Test: Let p be a critical number. If $f''(p) > 0$, then f has a local minimum at p; If $f''(p) < 0$, then f has a local maximum at p; If $f''(p) = 0$, then nothing.

Example 100 *Let* $f(x) = x^4 - 4x^3 + 4x^2 + 4$.

(c) Find all the local minimum points and all the local maximum points by Second Derivative Test.

(d) Find all the points of inflection.

(e) State intervals of concavity.

(f) Sketch the graph.

Example 101 *Consider the function*

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Example 102 *Use the first and second derivatives of* $f(x) = e^{1/x}$*, together with asymptotes, to sketch its graph.*

6.1-6.2 Applications of Derivatives

Maximum and Minimum Values

- Absolute (Global) Maximum and Minimum: $f(x)$ has a Global (Absolute) Maximum at p if $f(p) \geq f(x)$ for all x in the domain; $f(x)$ has a Global (Absolute) Minimum at p if $f(p) \leq f(x)$ for all x in the domain;
- Local (or relative) extrema: $f(x)$ has a local minimum at p if $f(p) \leq f(x)$ for points x near p; $f(x)$ has a local maximum at p if $f(p) \geq f(x)$ for points x near p;
- Critical number: A point p in the domain such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number, $(p, f(p))$ is a critical point, $f(p)$ is a critical value.

EXTREME VALUE THEOREM: If f(x) is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].
FERMAT's THEOREM: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition 17 *Let $c \in D(f)$. If $f'(c) = 0$ or $f'(c)$ is undefined, then c is called a critical number (or critical point).*

Example 103 *Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.*

First Derivative Test: Let c be a critical number. If f' changes from - to + at c, then f has a local minimum at c; If f' changes from + to - at c, then f has a local maximum at c.

Second Derivative Test: Let c be a critical number. If $f''(c) > 0$, then f has a local minimum at c; If $f''(c) < 0$ changes from + to - at c, then f has a local maximum at c; If $f''(c) = 0$, then the test provides no answer, go back to the first derivative test.

Example 104 *The Tradeoff between Medication and Site Effects: Suppose that a patient is given a dosage x of some medication, and the probability of a cure is*

$$P(x) = \frac{\sqrt{x}}{1+x}.$$

(a) Find the domain and the critical numbers.
(b) State the intervals of increase and decrease.
(c) Find the local maximum .

Example 105 *Spread of a Pollutant: The concentration of a pollutant (measured in ppm, parts per million) at a fixed location x units from the source, is given by*

$$c(t) = \frac{N}{\sqrt{4\pi kt}} e^{-x^2/4kt},$$

where $N, k, t > 0$. When does the pollution reach its max?

Example 106 *Let $g(x) = x + 2\sin x$, $0 \leq x \leq 2\pi$.*
(a) Find all the critical numbers.
(b) State all the intervals of increase and decrease.
(c) Find all the local minimum points and all the local maximum points.

Absolute max and min, CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ on a closed interval $[a, b]$:

1. Find all the critical numbers, e.g., x_1, ..., x_n.
2. global minimum = $\min\{f(x_1), ..., f(x_n), f(a), f(b)\}$;
   global maximum = $\max\{f(x_1), ..., f(x_n), f(a), f(b)\}$.

Example 107 *Find the global maximum and minimum of the function*

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Example 108 *An open cylinder container has surface area $3\pi ft^2$. What dimensions will maximize the volume?*

Example 109 *Strength of Bones: The total mass of a bone and the mass of the marrow can be modeled by*

$$f(m) = c(2 - m^2)(1 - m^4)^{-2/3}, \quad 0 \le m \le 1,$$

where $m = 0$ *characterizes a solid bone, and* $m = 1$ *describes a bone that is all marrow,* m *represents marrow cavity radius. When will the total mass reaches the minimum?*

6.4 L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad 0 \cdot \infty, \quad 0^0, \ldots$$

L'Hospital's rule: If $\frac{f(x)}{g(x)}$ becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to x_0$, where x_0 is finite or ∞, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Remark. $x \to x_0$ can be replaced by any of the symbols $x \to x_0^+$, $x \to x_0^-$, $x \to \infty$, or $x \to -\infty$.

Example 110 *Calculate*

$$\lim_{t \to 0} \frac{\sin x}{x}, \quad \lim_{t \to 0} \frac{\sin x}{x^2}, \quad \lim_{t \to 0} \frac{e^t - t - 1}{t^2}.$$

Example 111 *Calculate*

$$\lim_{x \to \infty} x^2 e^{-x}, \quad \lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x.$$

Example 112 *Calculate*

$$\lim_{t \to 0} x^{\sin x}.$$

6.5 Graphing Functions

The following checklist is intended as a guide to sketching a curve y = f(x) by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- A. DOMAIN
- B. INTERCEPTS
- C. SYMMETRY
  1. EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.
  2. ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.
  3. PERIODIC FUNCTION: f(x + p) = f(x) for all x in D, where p is a positive constant. The smallest such number p is called the period.
- D. ASYMPTOTES
  - HORIZONTAL: $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a HA.
  - VERTICAL: $\lim_{x\to a^\pm} f(x) = \pm\infty$, then $x = a$ is a VA.
  - SLANT: If $\lim_{x\to\infty}[f(x) - (mx + b] = 0$, y =mx + b is called a slant asymptote.
- E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.
- F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.
- G. CONCAVITY AND POINTS OF INFLECTION

Example 113 *Sketch the following functions:*

$$f(x) = x^4 + 8x^3 + 18x^2 + 1,$$

$$g(x) = \frac{2x^2}{x^2 - 4},$$
$$h(x) = xe^x,$$
$$k(x) = \frac{\ln x}{x^2}.$$

5.7 Approximating Functions with Polynomials

LINEAR APPROXIMATIONS: we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Definition 18 *The approximation*

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 114 *Find the linearization of the function* $f(x) = \sqrt{x}$ *at* $a = 9$ *and use it to approximate the numbers* $\sqrt{9.01}$*.*

Example 115 *The linearization of the function* $f(x) = \sin x$ *at* $a = 0$ *is L(x) = x.*

Taylor polynomial Approximation: nth-degree Taylor polynomial of $f(x)$ at a is defined as

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + cdots + \frac{f^{(n)}}{n!}(x - a)^n.$$

Example 116 *Find the 4th-degree Taylor polynomial of* $f(x) = e^x$ *at* $x = 0$ *and use it to approximate* $e^{0.1}$.

6.3 Reasoning about Functions

Intermediate Value Theorem: Let $f(x)$ be continuous on $[a, b]$ and K is a number between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ such that $f(c) = K$.

Example 117 *Show that $x^4 = 5x + 23$ has solution in [3,4].*

MEAN VALUE THEOREM: Let $f(x)$ be a function that satisfies the following two hypotheses:

1. $f(x)$ is continuous on the closed interval [a, b]
2. $f(x)$ is differentiable on the open interval (a, b)

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 118 *Suppose that f(0) = -3 and $f'(x) \leq 5$ for all values of x. How large can f(2) possibly be?*

6.6 Newton's Method

Sometimes we are presented with a problem which cannot be solved by simple algebraic means. For instance, if we needed to find the roots of the polynomial

$$x^3 - x + 1 = 0,$$

we would find that the tried and true techniques just wouldn't work. However, we will see that calculus gives us a way of finding approximate solutions.

Newton's method: To approximate solutions of the equation $f(x) = 0$, start from x_1, we have approximate solutions

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided we have started with a good value for x_1 , this will produce approximate solutions to any degree of accuracy.

Example 119 *Find the roots* $\sqrt[6]{2}$ *by Newton's method, correct to 8 decimals.*

Solution: Let $f(x) = x^6 - 2$. Then $\sqrt[6]{2}$ is a solution of the equation $f(x) = 0$. Note that $f(2) = 62 > 0$ and $f(0) = -2 < 0$. This tells us that the root is between 0 and 2. So we chose $x_1 = 1$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5} = \frac{5x_n^6 + 2}{6x_n^5}.$$

With our initial guess of $x_1 = 1$, we can produce the following values:

x_1	1
x_2	1.16666667
x_3	1.12644368
x_4	1.12249707
x_5	1.12246205
x_6	1.12246205

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to 8 decimal places.

Example 120 *Find the roots of the polynomial* $f(x) = x^3 - x + 1 = 0$ *by Newton's method with* $x_1 = -1$*, correct to 6 decimals.*

Solution: Note that $f(-2) = -5$ and $f(-1) = 1$. This tells us that the root is between -2 and -1. So we chose $x_1 = -1$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1} = \frac{2x_n^3 - 1}{3x_n^2 - 1}.$$

With our initial guess of $x_1 = -1$, we can produce the following values:

x_1	-1
x_2	-1.500000
x_3	-1.347826
x_4	-1.325200
x_5	-1.324718
x_6	-1.324717
x_7	-1.324717
x_8	-1.324717

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to six decimal places. In fact, this was obtained after only five relatively painless steps.

6.7 Stability of Discrete-Time Dynamical Systems

- An equilibrium x^* is stable if the graph of the updating function $f(x)$ crosses the diagonal $y = x$ from above to below, i.e., $f'(x^*) < 1$;
- An equilibrium x^* is unstable if the graph of the updating function $f(x)$ crosses the diagonal $y = x$ from below to above, i.e., $f'(x^*) > 1$.

Example 121 *Suppose the fraction x of mutant bacteria in a population of bacteria is given by the updating function*

$$f(x) = \frac{1.2x}{1.2x + 2(1-x)},$$

where 1.2 is the per capita production of the original type and 2 is the per capita production of the mutant type. Find the equilibria and analyze the stability.

6.8 The Logistic Dynamical Systems

Consider the logistic dynamical system

$$N_{t+1} = rN_t(1 - \frac{N_t}{K}).$$

$$\text{per capita production} = r(1 - \frac{N_t}{K}),$$

where N represents population size, r is the greatest possible production, and K is the capacity. Let

$$x_t = \frac{N_t}{K},$$

then

$$x_{t+1} = rx_t(1 - x_t).$$

The equilibria are $x^* = 0, 1 - \frac{1}{r}$.

r	x^*	stability
0.5	0	stable
1.5	0	unstable
1.5	1/3	stable
2.5	0	unstable
2.5	0.6	stable
3.5	0	unstable
3.5	5/7	unstable

Theorem 8 *(Stability Theorem). Consider the DTDS*

$$x_{t+1} = f(x_t)$$

with an equilibrium x^*. If $|f'(x^*)| < 1$, then x^* is stable; If $|f'(x^*)| > 1$, then x^* is unstable.

Example 122 *Concerning the above logistic dynamical system, $f'(0) = r$, $f'(1 - 1/r) = 2 - r$.*

7.1 Differential Equations

n-th order DE: $f(y^{(n)}, ..., y', y, t) = 0$.

- Pure-time differential equation: $\frac{df(t)}{dt} = F(t)$.
- Autonomous differential equations: $\frac{df(t)}{dt} = F(f)$.
- Nonautonomous differential equations: $\frac{df(t)}{dt} = F(f, t)$.

Some basic models:

- Exponential model: The rate of change of population growth is proportional to population size:
$$P'(t) = rP,$$
where r is a constant.
- Logistic model:
$$P'(t) = rP(1 - \frac{P}{K}),$$
where $r > 0$ is the relative growth rate, $L > 0$ is the capacity.
- Newton's Law of Cooling:
$$\frac{dT}{dt} = \alpha(A - T),$$
where α is a positive constant, A is a constant.

Euler's method

If h (or Δt) is the step size, $y' = F(t, y)$, $y(t_0) = y_0$, then

$$y_n = y_{n-1} + hF(t_{n-1}, y_{n-1}), \quad t_n = t_{n-1} + h.$$

Example 123 *Use Euler's Method to solve the following differential equation (keep THREE decimal places):*

$$\frac{df}{dt} = e^{-t}, \quad f(0) = 0.$$

(a) Use step size $\Delta t = 1$ *to find* $f(4)$.
(b) Use step size $\Delta t = 0.5$ *to find* $f(4)$.
(c) Which $f(4)$ *is better, if* $f(t) = 1 - e^{-t}$*?*

Example 124 *Consider the differential equation*

$$\frac{dy}{dt} = ky, \quad y(0) = C.$$

Use Euler's method to approximate the solution $y(t)$ *with step size* $\Delta t = \frac{t}{n}$.

7.2 Antiderivatives

Definition 19 *A function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I. We also call it the indefinite integral of $f(x)$.*

Some basic results:

function	antiderivative	formula
k	$kx + C$	$\int kdx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C;\ (n \neq -1)$
e^{kx}	$\frac{1}{k}e^{kx} + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$
a^{kx}	$\frac{a^{kx}}{k\ln a} + C$	$\int a^{kx}dx = \frac{a^{kx}}{k\ln a} + C$
$\frac{1}{x}$	$\ln\|x\| + C$	$\int \frac{1}{x}dx = \ln\|x\| + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$	$\int \cos kx dx = \frac{1}{k}\sin kx + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$	$\int \sin kx dx = -\frac{1}{k}\cos kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	$\int \sec^2 kx dx = \frac{1}{k}\tan kx + C$
$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	$\int \sec kx \tan kx dx = \frac{1}{k}\sec kx + C$
$\frac{1}{\sqrt{1-(kx)^2}}$	$\frac{1}{k}\arcsin kx + C$	$\int \frac{1}{\sqrt{1-(kx)^2}}dx = \frac{1}{k}\arcsin kx + C$
$\frac{1}{1+(kx)^2}$	$\frac{1}{k}\arctan kx + C$	$\int \frac{1}{1+(kx)^2}dx = \frac{1}{k}\arctan kx + C$
		$\int kf(x)dx = k\int f(x)dx$
		$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$

Example 125 $\int \frac{x^2-1}{x^3}dx = \int (x^{-1} - x^{-3})dx = \ln|x| + \frac{1}{2x^2} + C.$

$\int \sin 4x + e^{5x}dx = -\frac{1}{4}\cos 4x + \frac{1}{5}e^{5x} + C.$

Example 126 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}.$$

Example 127 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}, \quad f(1) = 3.$$

Example 128 *Let* $y = y(x)$ *be the solution of the differential equation*

$$\frac{dy}{dx} = \frac{\pi \sin \pi x + 4x}{9y^2}$$

subject to the initial condition $y(0) = 1$. *Find* $y(1)$.

Example 129 *The number of AIDS cases* $A(t)$, *where* t *is measured in years since 1981, is modeled by*

$$A(t) = 523.8t^2, \quad A(0) = 340 \textit{ people}.$$

Find $A(t)$.

7.3-7.4 Definite Integral and Area

Three ways to estimate the area of the region S bounded by the continuous function $y = f(x)$ (where $f(x) \geq 0$), $x = a$, $x = b$ and the x-axis:

We divide the interval $[a, b]$ into n equal parts with endpoints $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$,..., $x_n = a + \frac{n(b-a)}{n} = b$, $\Delta x = \frac{b-a}{n}$,

$$\begin{aligned} L_n &= \sum_{i=0}^{n-1} f(x_i)\Delta x = [f(x_0) + f(x_1) + ... + f(x_{n-1})]\, \Delta x, \\ R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = [f(x_1) + ... + f(x_{n-1}) + f(x_n)]\, \Delta x, \\ M_n &= \sum_{i=1}^{n} f(\frac{x_{i-1} + x_i}{2})\Delta x = \left[f(\frac{x_0 + x_1}{2}) + f(\frac{x_1 + x_2}{2}) + ... + f(\frac{x_{n-1} + x_n}{2}) \right] \Delta x. \end{aligned}$$

Here L_n is called Left-hand Sum, R_n is Right-hand Sum, M_n is called Midpoint Sum, or Midpoint Rule.

Example 130 *Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.*

Definition 20 *The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is:*

$$Area = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

Definition 21 *Definite integral = limit of Riemann sum:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} GRS,$$

where

$$GRS(\text{General Riemann Sum}) = \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \leq c_i \leq x_i, \Delta x = \frac{b-a}{n}.$$

The relation to area is:

$$\int_a^b f(x)dx = \text{area above } x\text{-axis - area below } x\text{-axis.}$$

Some basic properties about definite integral:

- $\int_a^b c dx = c(b-a)$;
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
- $\int_a^a f(x)dx = 0$;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- Constant multiple: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- Comparison of Definite Integrals: If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Example 131 *Let* $\int_1^5 f(x)\,dx = 3$, $\int_1^5 g(x)\,dx = 5$. *Calculate* $\int_1^5 [2f(x) - g(x) - 1]\,dx$.

The Fundamental Theorem of Calculus :

- **If** $F'(x) = f(x)$**, then**
$$\int_a^b f(x)dx = F(b) - F(a).$$
- **If**
$$g(x) = \int_a^x f(t)dt,$$
then $g'(x) = f(x)$**.**
- **General formula**
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

Example 132

$$\begin{aligned}
\frac{d}{dx}\int_0^x f(t)dt &= f(x), \\
\frac{d}{dx}\int_0^{x^2} f(t)dt &= \frac{d}{du}\int_0^u f(t)dt \cdot \frac{du}{dx} = 2xf(x^2), \quad u = x^2, \\
\frac{d}{dx}\int_{x^2}^{x^3} f(t)dt &= \frac{d}{dx}\left(\int_{x^2}^0 f(t)dt + \int_0^{x^3} f(t)dt\right) = -2xf(x^2) + 3x^2 f(x^3).
\end{aligned}$$

Example 133 ***Let*** $g(x) = \int_a^x e^{t^2} dt$***. Calculate*** $g'(2)$ ***and*** $g''(2)$***.***

Example 134 ***Calculate*** $\int_0^2 3^t dt$.

Example 135 ***Find the area of the region between the*** x***-axis and the graph of*** $f(x) = x^3 - 2x^2 - 3x$***,*** $-1 \leq x \leq 3$***.***

7.5 Substitution and Integration by Parts

Substitution

- **For indefinite integral:** $\int f(g(x))g'(x)dx = \int f(u)du, \quad u = g(x)$. **In the final result, we have to replace** u **by** $g(x)$**;**
- **For definite integral:** $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$

Example 136 ***Evaluate***

$$\int (2x-1)(x^2-x)^{100}\,dx.$$

Example 137

$$\int x\sqrt{x^2+1}\,dx.$$

Example 138

$$\int_0^1 x\sqrt{x^2+1}\,dx.$$

Example 139 ***Find***

$$\int \frac{1}{e^{-x}+1}dx.$$

Example 140 ***Evaluate***

$$\int x^2 e^{x^3+1}\,dx.$$

Example 141 ***Evaluate***

$$\int \frac{x}{\sqrt{1-x}}\,dx.$$

Example 142 ***Calculate***

$$\int \tan x\,dx.$$

Integration by Parts

Integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Example 143 ***Evaluate***

$$\int (x^2 + x + 1)e^x \, dx.$$

Example 144 ***Evaluate***

$$\int 4x^3 \ln x \, dx.$$

Example 145 ***Evaluate***

$$\int_0^1 \arctan x \, dx.$$

Example 146 ***Evaluate***

$$\int e^x \cos x \, dx.$$