MAT 2377 (Winter 2017)

Assignment 4

Deadline : Please submit in the dropbox at 585 King Edward before 7:00pm on Thursday 9 March, 2017.
There are 4 questions.

Please solve the following problems with a calculator authorized by the Faculty of Science (TI30, TI34, Casio fx-260 or Casio fx-300) :

1. We collect 25 water samples. Suppose that each water sample has a 4% probability of being polluted, and that the samples are mutually independent. Let X be the number of polluted samples among these 25 samples.

[1] (a) What is the probability distribution of X?

[3] (b) Calculate $P(X = 3)$ exactly, and also with the Poisson approximation.

[2] (c) Calculate $P(X > 2)$ and $P(4 \leq X < 6)$

Solution:

(a) X has binomial distribution with parameters $n = 25$ and $p = 0.04$ (note: it is also acceptable to give the probability mass function: $p_X(x) = \binom{25}{x} 0.04^x (1 - 0.04)^{-x}$ for $x = 0, 1, \ldots, 25$).

(b) Exact probability : $P(X = 3) = \binom{25}{3}(0.04)^3(0.96)^{22} = 0.05996$
Poisson Approximation : we have $\mu = n\,p = 1$, so $P(X = 3) \approx e^{-\mu}\frac{\mu^3}{3!} = 0.0613$

(c)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - \left[\binom{25}{0}(0.04)^0(0.96)^{25} + \binom{25}{1}(0.04)^1(0.96)^{24} + \binom{25}{2}(0.04)^2(0.96)^{23}\right] = 0.0765$$

and

$$P(4 \leq X < 6) = P(X = 4) + P(X = 5) = \binom{25}{4}(0.04)^4(0.96)^{21} + \binom{25}{5}(0.04)^5(0.96)^{20} = 0.01615$$

2. Consider the water samples from Question 1. Suppose that we collect the water samples one at a time.

[2] (a) What is the probability that the fifth water sample we collect will be the first polluted sample?

[2] (b) What is the probability that the sixth water sample we collect will be the second polluted sample?

[2] (c) What is the mean and standard deviation of the number of samples required to observe two polluted samples?

Solution:

(a) We want

$$P(T = 5) = (1 - p)^4\, p = 0.03397, \quad \text{where } T \sim \text{geometric with } p = 0.04$$

(b) We want

$$P(T_2 = 6) = \binom{5}{1} p^2 (1 - p)^4 = 0.00679, \quad \text{where } T_2 \sim \text{negative binomial with } r = 2 \text{ and } p = 0.04$$

(c) The mean and standard deviation are

$$E[T_2] = \frac{r}{p} = \frac{2}{0.04} = 50 \quad \text{et} \quad \sigma_{T_2} = \sqrt{V[T_2]} = \sqrt{\frac{r(1-p)}{p^2}} = \sqrt{\frac{2(0.96)}{(0.04)^2}} = 34.641$$

3. Electronic components are shipped in boxes of 18 components. Each component has a 10% probability of being defective, and the components are mutually independent. We check each box one at a time.

   (a) What is the probability that a given box has no defective components?

   (b) Consider 12 boxes. What is the probability that *at least* 3 of these boxes contain only non-defective components?

   (c) What is the probability distribution of the number of boxes that we must check until we find a box that contains only non-defective components?

   Solution:

   (a) We want
   $$P(T=5) = (1-p)^4\,p = 0.03397, \quad \text{where } T \sim \text{geometric with } p = 0{,}04.$$

   (b) We want
   $$P(T_2=6) = \binom{5}{1} p^2 (1-p)^4 = 0.00679, \quad \text{where } T_2 \sim \text{negative binomial with } r = 2 \text{ and } p = 0.04$$

   (c) The mean and standard deviation are respectively
   $$E[T_2] = \frac{r}{p} = \frac{2}{0.04} = 50 \quad \text{and} \quad \sigma_{T_2} = \sqrt{V[T_2]} = \sqrt{\frac{r(1-p)}{p^2}} = \sqrt{\frac{2(0.96)}{(0.04)^2}} = 34.641$$

4. Suppose that geomagnetic storms occur according to a Poisson process with parameter $\lambda = 2.5$ storms every 365 days. (These storms can interfere with power line networks and satellites, and are a cause for concern for power and telecommunications companies.)

   (a) Compute the probability that at least 4 geomagnetic storms will occur next year.

   (b) Compute the expected number of geomagnetic storms in the next 120 days.

   (c) Compute the probability that there will be zero geomagnetic storms during the 153 days between July 1st and December 1st.

   Solution: Let N_t be the number of geomagnetic storms in t days. N_t has poisson *distribution* with mean $\mu = \lambda\, t$.

   (a) We want
   $$P(N_{365} \geq 4) = 1 - P(N_{365} < 4) = 1 - e^{-\mu}\frac{\mu^0}{0!} - e^{-\mu}\frac{\mu^1}{1!} - e^{-\mu}\frac{\mu^2}{2!} - e^{-\mu}\frac{\mu^3}{3!} = 0.2424,$$
   where $\mu = \lambda t = 2.5 \frac{\text{storms}}{365\,\text{days}} \times 365\,\text{days} = 2.5\text{storms}$.

   (b) $E[N_{120}] = \lambda t = 2.5 \frac{\text{storms}}{365\,\text{days}} \times 120\,\text{days} = 0.821$ storms.

   (c) We want $P[N_{153/365} = 0] = e^{-\mu}\,\mu^0/0! = e^{-\mu} = 0.35066$,
   where $\mu = \lambda\, t = 2.5 \frac{\text{storms}}{365\,\text{days}} \times 153\,\text{days} = 1.04795$ storms.

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