

## MAT 2777 (Hiver 2017)

## Assignment 3 - solutions

*There are 3 questions*

1. (a) Here are the probability mass functions of  $X$  and  $Y$ .

$$p_X(x) = \begin{cases} -2, & 15/50 \\ 0, & 25/50 \\ 2, & 8/50 \end{cases} \quad \text{and} \quad p_Y(y) = \begin{cases} -5, & 17/50 \\ 0, & 15/50 \\ 5, & 18/50 \end{cases}$$

[1] So,

$$\mu_X = E[X] = \sum_{x \in R_X} x p_X(x) = (-2) \left(\frac{15}{50}\right) + 0 \left(\frac{25}{50}\right) + 2 \left(\frac{15}{50}\right) = \frac{-14}{50} = -0.28$$

[1] and

$$\mu_Y = E[Y] = \sum_{y \in R_Y} y p_Y(y) = (-5) \left(\frac{17}{50}\right) + 0 \left(\frac{15}{50}\right) + 5 \left(\frac{18}{50}\right) = \frac{5}{50} = 0.1.$$

[1] With the joint probability mass function, we can compute.

$$E[XY] = \sum_{x \in R_{XY}} x y p_{XY}(x, y) = 60/50 = 1.2.$$

[1] So the covariance of  $X$  and  $Y$  is

$$\text{Cov}[X, Y] = E[XY] - \mu_X \mu_Y = 1.2 - (-0.28)(0.1) = 1.228$$

[1] (b) Since  $\text{Cov}[X, Y] \neq 0$ , we have that  $X$  and  $Y$  are **not** independant.

(c) We have

$$E[X^2] = \sum_{x \in R_X} x^2 \times p_X(x) = (-2)^2 \left(\frac{15}{50}\right) + 0^2 \left(\frac{25}{50}\right) + 2^2 \left(\frac{15}{50}\right) = \frac{92}{50}$$

$$E[Y^2] = \sum_{y \in R_Y} y^2 \times p_Y(y) = (-5)^2 \left(\frac{17}{50}\right) + 0^2 \left(\frac{15}{50}\right) + 5^2 \left(\frac{18}{50}\right) = \frac{875}{50}$$

[1] and so,

$$V[X] = E[X^2] - \mu_X^2 = \frac{92}{50} - (-0.28)^2 = 1.7616;$$

[1]

$$V[Y] = E[Y^2] - \mu_Y^2 = \frac{875}{50} - (0.1)^2 = 17.49.$$

[2] Therefore,

$$V[X+2Y] = V[X] + V[2Y] + 2\text{Cov}[X, 2Y] = V[X] + 2^2 V[Y] + 2 \times 2 \text{Cov}[X, Y] = 1.7616 + 4(17.49) + 4(1.228) = 76.6336$$

[2] and

$$\begin{aligned} V[7X - Y] &= V[7X] + V[-Y] + 2\text{Cov}[7X, -Y] = 7^2 V[X] + (-1)^2 V[Y] + (2)(7)(-1)\text{Cov}[X, Y] \\ &= (49)(1.7616) + (1)(17.49) + (-14)(1.228) = 86.6164 \end{aligned}$$

[1] 2. (a)

$$P(X = -1) = p_{XY}(-1, -1) + p_{XY}(-1, 0) = \frac{1}{20} + \frac{4}{20} = \frac{5}{20} = 0.25$$

[1]

$$P(X = -1|Y = -1) = \frac{P(X = -1, Y = -1)}{P(Y = -1)} = \frac{p_{XY}(-1, -1)}{p_{XY}(-1, -1)} = 1.$$

[2] (b)

$$p_X(x) = \begin{cases} 1/20 + 4/20 = 5/20, & x = -1 \\ 2/20 + 7/20 + 2/20 = 11/20, & x = 0 \\ 3/20, & x = 1 \\ 1/20, & x = 2 \end{cases} \quad \text{and} \quad p_Y(y) = \begin{cases} 1/20, & y = -2 \\ 1/20, & y = -1 \\ 4/20 + 2/20 = 6/20, & y = 0 \\ 7/20 + 3/20 = 10/20, & y = 1 \\ 2/20, & y = 3 \end{cases}$$

[1] (c)

$$\mu_X = \sum_{x \in R_X} x p_X(x) = (-1) \left( \frac{5}{20} \right) + (0) \left( \frac{11}{20} \right) + (1) \left( \frac{3}{20} \right) + (2) \left( \frac{1}{20} \right) = 0$$

[1]

$$E[XY] = \sum_{(x,y) \in R_{XY}} xy \times p_{XY}(x,y) = (-1)(-1) \left( \frac{1}{20} \right) + 0 + 0 + 0 + 0 + (1)(1) \frac{3}{20} + 2(-2) \left( \frac{1}{20} \right) = 0$$

[1] and so,

$$\text{Cov}[X, Y] = E[XY] - \mu_X \mu_Y = 0 - 0 \times \mu_Y = 0 - 0 = 0$$

[2] (d) In part (a), we have  $P(X = -1) = \frac{1}{4} \neq 1 = P(X = -1|Y = -1)$ . So,  $X$  and  $Y$  are **not** independent.

Alternatively, you could have shown dependence by using the marginal probability mass functions you found in part (b):

$$p_X(2) p_Y(-2) = \frac{1}{20} \times \frac{1}{20} = \frac{1}{400} \neq \frac{1}{20} = p_{XY}(2, -2).$$

Therefore,  $p_X(x)p_Y(y)$  is not equal to  $p_{XY}(x,y)$  for all  $(x,y) \in R_X \times R_Y$ . Therefore,  $X$  and  $Y$  are **not** independent.

3. (a) We will use the fact that  $\sum_{x=0}^{\infty} (1/2)^x = 1/(1 - 1/2) = 2$  and  $\sum_{y=0}^{\infty} (1/2)^y = 1/(1 - 1/2) = 2$ . Solve

$$\begin{aligned} 1 &= \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} c (1/2)^x (1/2)^y = c \sum_{x=0}^{\infty} (1/2)^x \sum_{y=0}^{\infty} (1/2)^y = c \sum_{x=0}^{\infty} (1/2)^x (2) \\ &= 2c \sum_{x=0}^{\infty} (1/2)^x = 2c(2) = 4c. \end{aligned}$$

Therefore,  $c = 1/4$ .

(b) Let  $x$  be a number in the set  $\{0, 1, 2, \dots\}$ . We want

$$\begin{aligned} p_X(x) = P(X = x) &= \sum_{y \in R_Y} p_{XY}(x, y) = \sum_{y=0}^{\infty} (1/4) (1/2)^x (1/2)^y = (1/4) (1/2)^x \sum_{y=0}^{\infty} (1/2)^y \\ &= (1/4) (1/2)^x 2 = (1/2) (1/2)^x. \end{aligned}$$

Let  $y$  be a number in the set  $\{0, 1, 2, \dots\}$ . We want

$$\begin{aligned} p_Y(y) = P(Y = y) &= \sum_{x \in R_X} p_{XY}(x, y) = \sum_{x=0}^{\infty} (1/4) (1/2)^x (1/2)^y = (1/4) (1/2)^y \sum_{x=0}^{\infty} (1/2)^x \\ &= (1/4) (1/2)^y 2 = (1/2) (1/2)^y. \end{aligned}$$

(c) For all  $(x, y) \in R_X \times R_Y$ , we have

$$p_{XY}(x, y) = (1/4)(1/2)^x(1/2)^y = p_X(x) p_Y(y).$$

Therefore,  $X$  and  $Y$  are independent.

(d) Since  $X$  and  $Y$  are independent, we have that  $\text{Cov}(X, Y) = 0$ .

[ /20]