**MAT 2377 (Winter 2017)**

**Assignment 2**
**Solution**
*There are 5 questions*

**Remark:** *Please only grade questions 1, 4 and 5.*

[1] 1. First, we determine the probability mass function

$$p_X(x) = \begin{cases} 0.17; & x = 0 \\ 0.36; & x = 1 \\ 0.31; & x = 2 \\ 0.13; & x = 3 \\ 0.03; & x = 4 \end{cases}$$

[1] (a) We want to find $P(X < 2) = P(X = 0) + P(X = 1) = 0.17 + 0.36 = 0.53.$
[1] (b) We want to find $P(X > 3) = P(X = 4) = 0.03.$
[1] (c) We want to find $P(X = 1) = 0.36$
[1] (d) We want to find

$$E(X) = \sum_{x \in R_X} x\, p_X(x) = 0(0.17) + 1(0.36) + \cdots + 5(0.03) = 1.49.$$

**N.B.** For parts (a), (b) and (c) we may use the cumulative distribution function $F_X$ as follows:
(a) $P(X < 2) = P(X \leq 1) = F_X(1) = 0.53.$
(b) $P(X > 3) = 1 - P(X \leq 3) = 1 - F_X(3) = 1 - 0.97 = 0.03$
(c) $P(X = 1) = P(X \leq 1) - P(X \leq 0) = F_X(1) - F_X(0) = 0.53 - 0.17 = 0.36$

2. (a) $P(X = 6) = 0.3, \qquad P(X \leq 7) = P(X = 4) + P(X = 6) = 0.7,$
and $P(X > 5) = 1 - P(X \leq 5) = 1 - P(X = 4) = 0.6$
(b)We want to find

$$E[7 + 10\, X] = \sum_{x \in R_X} (7 + 10\, x)\, p_X(x) = (7 + 10(4))(0.4) + \cdots + (7 + 10(10))(0.1) = 67.$$

**Alternatively:** one can calculate

$$E[X] = 4(0.4) + 6(0.3) + 8(0.2) + 10(0.1) = 6.$$

Hence, $E[7 + 10\, X] = 7 + 10\, E[X] = 7 + 10(6) = 67.$
(c) We want to find

$$E[1/X] = \sum_{x \in R_X} (1/x) p_X(x) = (1/4)(0.4) + \cdots + (1/10)(0.1) = 0.185.$$

3. (a)The probability density function, $f_X(x)$, of the continuous random variable $X$ can be found as follows:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{(4)81}{x^5} = \frac{324}{x^5}, \quad \text{for} \quad x > 3.$$

(b) We need to evaluate
(i) $P(5 < X < 10) = F_X(10) - F_X(5) = \left(1 - \frac{81}{10^4}\right) - \left(1 - \frac{81}{5^4}\right) = 0.1215$

(ii) $P(5 < X < 10) = \int_5^{10} f_X(x)\,dx = \int_5^{10} (324/x^5)\,dx = \left[324/(-4x^4)\right]_5^{10} = 0.1215$

(c) The definition of the expected value of the continuous random variable $X$ leads to

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_3^{\infty} x\,(324/x^5)\,dx = \int_3^{\infty} 324/x^4\,dx = \left.\frac{324}{-3x^3}\right|_3^{\infty} = 0 - \frac{324}{-81} = 4.$$

(d) First, we compute

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f_X(x)\,dx = \int_3^{\infty} x^2\,(324/x^5)\,dx = \int_3^{\infty} 324/x^3\,dx = \left.\frac{324}{-2x^2}\right|_3^{\infty} = 0 - \frac{324}{-18} = 18.$$

Therefore, the standard deviation of $X$ is $\sigma_X = \sqrt{V[X]} = \sqrt{E[X^2] - \mu_X^2} = \sqrt{18 - 4^2} = 1.4142$.

**[1]** 4. (a)

$$1 = \sum_{x \in R_X} p_X(x) = 11\,p \quad \Rightarrow \quad p = \frac{1}{11}.$$

**[1]** (b)By definition, the expected value of the discrete random variable $X$ is

$$\mu_X = E\,[X] = \sum_{x \in R_X} x\,p_X(x) = 0(1/11) + 1(2/11) + 2(3/11) + 3(5/11) = \frac{23}{11} = 2.0909.$$

**[2]** (c) First, we calculate

$$E(X^2) = \sum_{x \in R_X} x^2\,p_X(x) = 0^2(1/11) + 1^2(2/11) + 2^2(3/11) + 3^2(5/11) = \frac{59}{11} = 5.3636.$$

Therefore, the standard deviation of $X$ is $\sigma_X = \sqrt{V[X]} = \sqrt{E[X^2] - \mu_X^2} = \sqrt{5.3636 - (2.0909)^2} = 0.99586$.

**[1]** (d) $E[100 - 10\,X] = 100 - 10\,E[X] = 100 - 10\,(2.0909) = 79.091$

**[1]** (e) We have $V[100 - 10\,X] = (-10)^2\,V[X]$. Hence, the standard deviation of $100 - 10\,X$ is $\sqrt{V[100 - 10\,X]} = |-10|\sqrt{V[X]} = 10\sigma_X = 10(0.99586) = 9.9586$.

**[1]** 5. (a)

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^{\infty} \frac{c}{x^3} dx = \left.\frac{c}{-2x^2}\right|_1^{\infty} = \frac{1}{2} \quad \Rightarrow \quad c = 2.$$

**[1]** (b) We need to find

$$P(X \le 10) = \int_1^{10} \frac{2}{x^3}\,dx = \left.\frac{2}{-2x^2}\right|_1^{10} = 1 - \frac{1}{100} = 0.99.$$

**[1]** (c) We need to find

$$P(X \le 2.5) = \int_1^{2.5} \frac{2}{x^3}\,dx = \left.\frac{2}{-2x^2}\right|_1^{2.5} = 1 - \frac{1}{(2.5)^2} = 0.84.$$

**[1]** (d) We need to find

$$P(X \le 2.5 | X \le 10) = \frac{P(\{X \le 2.5\} \cap \{X \le 10\}}{P(X \le 10)} = \frac{P(X \le 2.5)}{P(X \le 10)} = \frac{0.84}{0.99} = 0.8485.$$

[ **/15**]