[**7**] **1)** Solve the differential equation $y' = \dfrac{x^2 + xy}{xy + y^2}$.

Solution:
The equation is homogeneous, so $u = \dfrac{y}{x} \Rightarrow$
$u + xu' = y' = \dfrac{1+u}{u+u^2} = \dfrac{1}{u} \Rightarrow xu' = \dfrac{1}{u} - u = \dfrac{1-u^2}{u} \Rightarrow \dfrac{u}{1-u^2}u' = \dfrac{1}{x} \Rightarrow$
$-\dfrac{1}{2}\ln|1-u^2| = \ln|x| + c_1 \Rightarrow \ln|1-u^2| = -2\ln|x| + c_2 \Rightarrow 1 - u^2 = c_3 x^{-2}$
$\Rightarrow x^2 - y^2 = c_3$.

[**6**] **2)** Solve the differential equation $y'' + 3y' + 2y = 0,\ y(0) = 1,\ y'(0) = 2$.

Solution:
$y = e^{rx} \Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0 \Rightarrow r = -2, -1 \Rightarrow$
$y = c_1 e^{-2x} + c_2 e^{-x}$.
$y(0) = 1 \Rightarrow c_1 + c_2 = 1,\ y'(x) = -2c_1 e^{-2x} - c_2 e^{-x},\ y'(0) = 2 \Rightarrow$
$-2c_1 - c_2 = 2 \Rightarrow c_1 = -3$ and $c_2 = 4$. Thus, $y = -3e^{-2x} + 4e^{-x}$.

[**7**] **3)** Solve the Bernoulli equation $y' + \dfrac{y}{x} = 3xy^2$.

Solution:
$u = y^{-1} \Rightarrow y = u^{-1} \Rightarrow y' = -u^{-2}u' \Rightarrow -u^{-2}u' + x^{-1}u^{-1} = 3xu^{-2} \Rightarrow$
$u' - x^{-1}u = -3x,\ I(x) = e^{\int -x^{-1}dx} = e^{-\ln|x|} = |x|^{-1}$, and we may take $I(x) = x^{-1}$. The equation then becomes $x^{-1}u' - x^{-2}u = -3$, i.e.,
$(x^{-1}u)' = -3$. Thus, $x^{-1}u = -3x + c \Rightarrow u = -3x^2 + cx \Rightarrow$
$y = u^{-1} = \dfrac{1}{cx - 3x^2}$.

[**5**] **4.a)** Verify that the differential equation

$$(e^x \sin y + 2x)dx + (e^x \cos y + 2y)dy = 0$$

is exact and solve it.

Solution:
$P = e^x \sin y + 2x, Q = e^x \cos y + 2y,\ P_y = e^x \cos(y) = Q_x \Rightarrow$ the equation is exact.
$f_x = P = e^x \sin(y) + 2x \Rightarrow f(x,y) = e^x \sin(y) + x^2 + g(y),\ f_y = Q \Rightarrow$

$e^x\cos(y) + g'(y) = e^x\cos(y) + 2y \;\Rightarrow\; g'(y) = 2y \;\Rightarrow\; g(y) = y^2 + c \;\Rightarrow$ $f(x,y) = e^x\sin(y) + x^2 + y^2 + c$, and the general solution is $e^x\sin(y) + x^2 + y^2 = k$.

[5] **4.b)** Show that the differential equation $(4x + 3y^2)\;dx + (2xy)\;dy = 0$ is not exact. Find an integrating factor of the form $I = I(x)$ that makes it exact. Write down the new exact differential equation.
(DO NOT SOLVE THE NEW DIFFERENTIAL EQUATION.)

Solution:
$P = 4x + 3y^2,\; Q = 2xy,\; P_y = 6y,\; Q_x = 2y,\; P_y \neq Q_x$, so the equation is not exact. $\dfrac{P_y - Q_x}{Q} = \dfrac{4y}{2xy} = \dfrac{2}{x}$ is a function of x only, so there exists an integrating factor $I(x)$ given by $\dfrac{I'(x)}{I(x)} = \dfrac{2}{x}$, which gives $I(x) = x^2$. Thus, the new equation is $(4x^3 + 3x^2y^2) + 2x^3yy' = 0$, and is exact.

[7] **5)** Solve the differential equation $\quad xy' + (2 + 3x)y = xe^{-3x}$.

Solution:
The equation is linear, $y' + \left(\dfrac{2}{x} + 3\right)y = e^{-3x}$, with the integrating factor $I(x) = e^{\int\left(\frac{2}{x}+3\right)dx} = e^{2\ln|x|+3x} = x^2e^{3x}$. Then $(x^2e^{3x}y)' = x^2 \;\Rightarrow$
$x^2e^{3x}y = \dfrac{x^3}{3} + c \;\Rightarrow\; y = \left(\dfrac{x}{3} + \dfrac{c}{x^2}\right)e^{-3x}$.

[7] **6)** Solve the differential equation $\quad 9x^2y'' + 15xy' + 5y = 0,\; x > 0$.

Solution:

The equation is Euler, so $y = x^r \;\Rightarrow\; 9r(r-1) + 15r + 5 = 0 \;\Rightarrow$
$9r^2 + 6r + 5 = 0 \;\Rightarrow\; r = \dfrac{-6 \pm \sqrt{36 - 180}}{18} = -\dfrac{1}{3} \pm \dfrac{2}{3}i \;\Rightarrow$
$y = x^{-\frac{1}{3}}\left[c_1\cos\left(\dfrac{2}{3}\ln x\right) + c_2\sin\left(\dfrac{2}{3}\ln x\right)\right]$.

[15] **7)** Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ **b)** $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$ **c)** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt[3]{n}}{n+1}$

Solution:

(a) $|a_n|^{\frac{1}{n}} = \frac{2n}{n^2} = \frac{2}{n} \to 0 < 1$ as $n \to \infty$, hence the series converges by the root test.

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}}{(n+1)^e}\frac{n^e}{e^n} = e\left(\frac{n}{n+1}\right)^e \to e > 1$ as $n \to \infty$, so the series diverges by the ratio test.

(c) The series is alternating, with $b_n = \frac{\sqrt[3]{n}}{n+1}$, which is positive and decreasing, and $b_n \to 0$ as $n \to \infty$. Hence the series converges by the alternating series test. The series does not converge absolutely because $|a_n| = b_n > \frac{\sqrt[3]{n}}{n+n} = \frac{1}{2n^{2/3}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges since it is a p-series with $p \leq 1$. Thus, the series converges conditionally.

[**10**] **8)** Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}.$$

Solution:

$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)4^{n+1}}{n4^n} = 4\lim_{n\to\infty}\frac{n+1}{n} = 4.$

$-4 < x+2 < 4 \ \Rightarrow\ -6 < x < 2$. At $x = 2$, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, and at $x = -6$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. Thus, $I = [-6, 2)$.

[**7**] **9)** Solve the differential equation $y'' + 4y = 3\cos(2x)$.

Solution:

$y'' + 4y = 0,\ y = e^{rx} \ \Rightarrow\ r^2 + 4 = 0 \ \Rightarrow\ r = \pm 2 \ \Rightarrow$

$y_h = c_1\cos(2x) + c_2\sin(2x)$. Since $C\cos(2x) + D\sin(2x)$ satisfies the homogeneous equation, a particular solution of the nonhomogeneous equation must take the form $y_p = x[C\cos(2x) + D\sin(2x)]$. Then

$y_p' = [C\cos(2x) + D\sin(2x)] + x[-2C\sin(2x) + 2D\cos(2x)]$,

$y_p'' = -4C\sin(2x) + 4D\cos(2x) + x[-4C\cos(2x) - 4D\sin(2x)]$,
so $y_p'' + 4y_p = 3\cos(2x) \Rightarrow -4C\sin(2x) + 4D\cos(2x) = 3\cos(2x) \Rightarrow$
$C = 0,\ D = \dfrac{3}{4} \Rightarrow y_p = \dfrac{3}{4}x\sin(2x)$, and $y = y_h + y_p$.

[8] **10)** Find the Taylor series for the function $f(x) = \cos x$ at $a = \pi/3$.

Solution:
$f(x) = \cos(x),\ f'(x) = -\sin(x),\ f''(x) = -\cos(x),\ f'''(x) = \sin(x)$, etc.
Thus, $f(\pi/3) = \dfrac{1}{2},\ f'(\pi/3) = -\dfrac{\sqrt{3}}{2},\ f''(\pi/3) = -\dfrac{1}{2}, f'''(\pi/3) = \dfrac{\sqrt{3}}{2}$, etc., so that $f^{(2k)}(\pi/3) = \dfrac{(-1)^k}{2}$ and $f^{(2k+1)}(\pi/3) = \dfrac{(-1)^{k+1}\sqrt{3}}{2},\ k \geq 0$.
Thus, the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/3)}{n!}\left(x - \frac{\pi}{3}\right)^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{2(2k)!}\left(x - \frac{\pi}{3}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\sqrt{3}}{2(2k+1)!}\left(x - \frac{\pi}{3}\right)^{2k+1}.$$

[7] **11)** Use the binomial series to expand the function $\dfrac{1}{\sqrt[3]{8+x}}$ as a power series.

Solution:

$$\frac{1}{\sqrt[3]{8+x}} = (8+x)^{-1/3} = \frac{1}{2}\left(1 + \frac{x}{8}\right)^{-1/3} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/3}{n}\left(\frac{x}{8}\right)^n.$$

For $n \geq 1$, $\dbinom{-1/3}{n} = \dfrac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3} - 1\right)\left(-\frac{1}{3} - 2\right)\cdots\left(-\frac{1}{3} - (n-1)\right)}{n!}$

$$= \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(-\frac{3n-2}{3}\right)}{n!} = \frac{(-1)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n!} \Rightarrow$$

$$\frac{1}{\sqrt[3]{8+x}} = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n! 8^n}x^n\right].$$

[4] **12.a)** Suppose that the differential equation $x^2y'' + xy' + x^2y = 0$ has a power series solution of the form $y = \sum\limits_{n=0}^{\infty} c_n x^n$. Find a **recurrence relation** that determines c_n, $n = 0, 1, 2, 3, \ldots$.
(Do not solve the differential equation)

Solution:

$$y = \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=0}^{\infty} nc_n x^{n-1},\ y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} \Rightarrow$$

$\sum\limits_{n=0}^{\infty} n(n-1)c_n x^n + \sum\limits_{n=0}^{\infty} nc_n x^n + \sum\limits_{n=0}^{\infty} c_n x^{n+2} = 0 \Rightarrow \sum\limits_{n=0}^{\infty} n^2 c_n x^n + \sum\limits_{n=0}^{\infty} c_n x^{n+2} = 0 \Rightarrow$
$c_1 x + \sum\limits_{n=2}^{\infty} n^2 c_n x^n + \sum\limits_{n=0}^{\infty} c_n x^{n+2} = 0 \Rightarrow c_1 x + \sum\limits_{n=0}^{\infty} [(n+2)^2 c_{n+2} + c_n] x^n = 0 \Rightarrow$
$c_1 = 0$ and $(n+2)^2 c_{n+2} + c_n = 0$ for all $n \geq 0$, i.e., $c_{n+2} = \dfrac{-c_n}{(n+2)^2}$.

[**5**] **12.b)** Use the recurrence relation

$$c_1 = c_2 = 0, \quad c_{n+1} = \frac{c_{n-2}}{n+1} \text{ for } n = 2, 3, 4, ...$$

to solve the differential equation $y' - x^2 y = 0$, by power series solution, where $y = \sum\limits_{n=0}^{\infty} c_n x^n$.

Solution:
$n = 2 \Rightarrow c_3 = \dfrac{c_0}{3}$, $n = 3 \Rightarrow c_4 = \dfrac{c_1}{4} = 0$, $n = 4 \Rightarrow c_5 = \dfrac{c_2}{5} = 0$,
$n = 5 \Rightarrow c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{2!3^2}$, $n = 6 \Rightarrow c_7 = \dfrac{c_4}{7} = 0$, $n = 7 \Rightarrow c_8 = \dfrac{c_5}{8} = 0$,
$n = 8 \Rightarrow c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 2!3^2} = \dfrac{c_0}{3!3^3}$, $n = 9 \Rightarrow c_{10} = \dfrac{c_7}{10} = 0$,
$n = 10 \Rightarrow c_{11} = \dfrac{c_8}{11} = 0$, $n = 11 \Rightarrow c_{12} = \dfrac{c_9}{12} = \dfrac{c_0}{12 \cdot 3!3^3} = \dfrac{c_0}{4!3^4}$, etc.,
so $c_{3k+1} = c_{3k+2} = 0$, and $c_{3k} = \dfrac{c_0}{k!3^k}$ for $k \geq 0$. Thus,
$y = \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{k=0}^{\infty} c_{3k} x^{3k} = c_0 \sum\limits_{k=0}^{\infty} \dfrac{1}{k!3^k} x^{3k} = c_0 \sum\limits_{k=0}^{\infty} \dfrac{1}{k!} \left(\dfrac{x^3}{3}\right)^k = c_0 e^{\frac{x^3}{3}}$, as can be confirmed easily, since the equation is separable!