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## Review: Algebraic Manipulations.

• Simplify: Ex.  $\frac{\sqrt{x^{\frac{1}{2}} y^5}}{x^5 y^{\frac{1}{4}}} = \frac{(x^{\frac{1}{2}} y^5)^{\frac{1}{2}}}{x^5 y^{\frac{1}{4}}} = \frac{x^{\frac{1}{4}} y^{\frac{5}{2}}}{x^5 y^{\frac{1}{4}}}$

$$= x^{\frac{1}{4}-5} y^{\frac{5}{2}-\frac{1}{4}} = x^{-\frac{19}{4}} y^{\frac{9}{4}}$$

• Rationalize the denominator: Ex.  $\frac{1}{\sqrt{10}-3} = \frac{\sqrt{10}+3}{(\sqrt{10}-3)(\sqrt{10}+3)}$

$$= \frac{\sqrt{10}+3}{10-9} = \sqrt{10}+3$$

Ex.  $\frac{\sqrt{6}-\sqrt{8}}{\sqrt{6}+\sqrt{8}} = \frac{(\sqrt{6}-\sqrt{8})(\sqrt{6}-\sqrt{8})}{(\sqrt{6}+\sqrt{8})(\sqrt{6}-\sqrt{8})} = \frac{(\sqrt{6}-\sqrt{8})^2}{6-8} = \frac{(\sqrt{6}-\sqrt{8})^2}{-2}$

• Solve for  $x$ : ★ Idea: regard  $x$  as a variable but regard others as constants.

Ex.  $\frac{1}{x} + \frac{2}{y} = \frac{5}{z} \Rightarrow \frac{1}{x} = \frac{5}{z} - \frac{2}{y} = \frac{5y-2z}{zy}$

$$\Rightarrow zy = x(5y-2z) \Rightarrow x = \frac{zy}{5y-2z}$$

• Absolute value

Definition:  $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Since the definition is in two pieces, equations with absolute values often require two cases.

Ex.  $|x-3|=6$

There are two different cases by definition.

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$$x-3 \geq 0 \Rightarrow x \geq 3 \quad \text{or} \quad x-3 < 0 \Rightarrow x < 3$$

Consider the first case:

$$\text{If } x \geq 3, \text{ then } |x-3| = x-3 = 6 \Rightarrow x = 9.$$

Now we need to check that whether  $x=9$  falls into our assumption  $x \geq 3$ . Indeed,  $x=9 \geq 3$ . Therefore,  $x=9$  is a solution.

$$\text{If } x < 3, \text{ then } |x-3| = -(x-3) = 3-x = 6 \Rightarrow x = -3.$$

Again, check:  $x = -3 < 3$  ( $\checkmark$ ). So  $x = -3$  is also a solution.


• The same ideas work with inequalities:

Ex  $|x^2 - 12| > 3.$

Split into two cases:

$$(a) \text{ If } x^2 - 12 \geq 0 \Leftrightarrow x^2 \geq 12 \Leftrightarrow x > 2\sqrt{3} \text{ or } x < -2\sqrt{3}$$

$$\text{Then } |x^2 - 12| = x^2 - 12 > 3 \Rightarrow x^2 > 15 \Rightarrow x > \sqrt{15} \text{ or } x < -\sqrt{15}$$

Remember to check: 

$$\text{Indeed, } -\sqrt{15} < -\sqrt{12} \text{ and } \sqrt{15} > \sqrt{12}$$

Therefore,  $x > \sqrt{15}$  or  $x < -\sqrt{15}$  is part of the solution.

$$(b) \text{ If } x^2 - 12 < 0 \Leftrightarrow -\sqrt{12} < x < \sqrt{12}$$

$$\text{Then } |x^2 - 12| = -(x^2 - 12) = 12 - x^2 > 3 \Rightarrow 9 > x^2 \Rightarrow -3 < x < 3$$

Again, check  $-\sqrt{12} < -3 < x < 3 < \sqrt{12}$ . Therefore,  $-3 < x < 3$  is also a part of the solution.

$$\text{The whole solution set: } \{x < -\sqrt{15}\} \cup \{-3 < x < 3\} \cup \{x > \sqrt{15}\}$$

• Note that when we solve inequalities, sometimes, the question naturally requires to split into different cases to consider.

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(There is no "absolute" symbol in the inequalities).

Ex.  $\frac{3}{x} + 4 < -1 \Rightarrow \frac{3}{x} < -5$

If  $x > 0$ , then  $\frac{3}{x} < -5 \Rightarrow 3 < -5x \Rightarrow -\frac{3}{5} > x$

Now check:  $x < -\frac{3}{5}$  is incompatible with  $x > 0$ . Therefore,

$x < -\frac{3}{5}$  is not part of the solution.

If  $x < 0$ , then  $3 > -5x \Rightarrow x > -\frac{3}{5}$

Again check:  $-\frac{3}{5} < x < 0$ . ( $\checkmark$ ).

Therefore,  $-\frac{3}{5} < x < 0$  is our solution.