Short-Answer Questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

3 marks

1. (a) How long would it take an investment to triple if the interest rate is 6% compounded continuously?

Answer: $\dfrac{\ln 3}{0.06}$

Solution: **Marking scheme:** 1pt for proper FV formula; 1pt for subbing all correct values

$$FV = PVe^{it} \Rightarrow 3 = e^{t0.06} \Rightarrow t = \frac{\ln 3}{0.06}$$

3 marks

(b) An investment of \$15,000 grew to \$20,000 over two and half years. What was the monthly compounded nominal interest rate?

Answer: $i = 12 \cdot \left[\left(\frac{4}{3}\right)^{1/30} - 1\right]$

Solution: **Marking scheme:** 1pt for proper FV formula; 1pt for subbing in all proper values

$$FV = PV\left(1 + \frac{i}{n}\right)^{nt}$$

$$20,000 = 15,000\left(1 + \frac{i}{12}\right)^{12 \cdot 5/2}$$

$$\frac{20}{15} = \left(1 + \frac{i}{12}\right)^{30}$$

$$\left(\frac{4}{3}\right)^{1/30} = 1 + \frac{i}{12}$$

$$\left(\frac{4}{3}\right)^{1/30} - 1 = \frac{i}{12}$$

3 marks

(c) How much money is required to generate \$10,000 in interest over one year if the interest rate is 8% compounded continuously?

Answer: $\dfrac{10,000}{e^{0.08} - 1}$

Solution: **Marking scheme:** 1pt for proper 1st line; 1pt for factoring out PV properly

$$FV = PVe^{it}$$

$$PV + 10,000 = PVe^{0.08}$$

$$10,000 = PVe^{0.08} - PV$$

$$10,000 = PV[e^{0.08} - 1]$$

3 marks

2. (a) Find the value of the parameter c that makes the following function continuous at $x = 3$.

$$f(x) = \begin{cases} 2 + \sqrt{x+1}, & x < 3 \\ e^{cx}, & x \geq 3. \end{cases}$$

Answer: $c = \frac{\ln 4}{3}$

Solution: **Marking scheme:** 1pt for stating continuity def; 1pt for a correct work less an error

We need $\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} 2 + \sqrt{x+1} = 4$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} e^{cx} = e^{3c} = f(3).$$

Setting the two equal to each other, we solve

$$4 = e^{3c} \iff \ln 4 = 3c \iff c = \frac{\ln 4}{3}.$$

3 marks

(b) Use the IVT to show that $\ln(x) + 2^x = 1 + 4x$ has a solution.

Solution: *Solutions will vary:* **Marking scheme:** 1pt for defining what $f(x)$ is; 1pt for finding the interval; 1pt for stating why we can use the IVT - i.e. continuity

Set $f(x) = \ln(x) + 2^x - 4x - 1$. We need to show $f(x) = 0$ somewhere, or has a sign change. This function is continuous on its domain $(0, \infty)$. We plug in to find

$$f(1) = 0 + 2 - 4 - 1 = -3,$$
$$f(5) = \ln 5 + 2^5 - 20 - 1 = 11 + \ln 5 > 0.$$

Thus, the IVT guarantees the existence of a solution between 1 and 5.

3 marks 3. (a) Evaluate $\lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2}$.

Answer: $\frac{1}{3}$

Solution: **Marking scheme:** 1pt for writing the correct conjugate, 1pt for cancelling out the factor

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} &= \lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} \\
&= \lim_{x\to 2} \frac{\sqrt{2x+5}-3}{x-2} \cdot \frac{\sqrt{2x+5}+3}{\sqrt{2x+5}+3} \\
&= \lim_{x\to 2} \frac{2x+5-9}{x-2} \cdot \frac{1}{\sqrt{2x+5}+3} \\
&= \lim_{x\to 2} \frac{2(x-2)}{(x-2)} \cdot \frac{1}{\sqrt{2x+5}+3} \\
&= \frac{2}{(\sqrt{9}+3)} \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

3 marks (b) Given that $f(1)=2$, $f'(1)=-1$, and $p(x)=\frac{f(x)}{x^2+1}$. What is the equation of the tangent line to the curve $y=p(x)$ at $x=1$?

Answer: $y = 1 - \frac{3}{2}(x-1)$
or $y = -\frac{3}{2}x + \frac{5}{2}$.

Solution: **Marking scheme:** 1pt for p'(x); 1pt for both p'(1) and p(1)
We'll need the derivative of $p(x)$ to get the slope of the tangent line:

$$p'(x) = \frac{f'(x)(x^2+1) - 2xf(x)}{(x^2+1)^2}.$$

At $x=1$, $p'(1) = -\frac{3}{2}$. Now we just need the y-value when $x=1$: $p(1) = \frac{f(1)}{1^2+1} = 1$.
The equation for the tangent line can now be written out.

3 marks (c) Compute y' given $y = \cos\left(e^{x\sin x}\right)$.

Answer: $-\sin\left(e^{x\sin x}\right) \cdot e^{x\sin x} \cdot (\sin x + x\cos x)$

Solution: **Marking scheme:** 1pt for each correct layer. i.e -1 pt for each error
By the chain rule:

$$\begin{aligned}
y &= \cos\left(e^{x\sin x}\right) \\
y' &= -\sin\left(e^{x\sin x}\right) \cdot e^{x\sin x} \cdot (\sin x + x\cos x)
\end{aligned}$$

3 marks 4. (a) Determine the slope of the tangent line to the curve

$$y^3 + x^2y^4 = 1 + 2x$$

at the point $(0, 1)$.

Answer: $m = \frac{2}{3}$

Solution: **Marking scheme:** 1pt for derivative; 1pt for isolating y' or subbing in
We need y' at the point $(0, 1)$.

$$\begin{aligned} y^3 + x^2y^4 &= 1 + 2x \\ 3y^2y' + 2xy^4 + x^2 \cdot 4y^3y' &= 2 \\ (3y^2 + x^2 \cdot 4y^3)y' &= 2 - 2xy^4 \\ y' &= \frac{2 - 2xy^4}{3y^2 + x^2 \cdot 4y^3} \Longrightarrow m = \frac{2}{3} \end{aligned}$$

3 marks (b) Find the value of the parameters A and B that make the following function differentiable at $x = 1$.

$$f(x) = \begin{cases} Ax + B, & x < 1 \\ \ln x, & x \geq 1 \end{cases}$$

Answer: $A = 1, B = -1$

Solution: **Marking scheme:** 2pt for arguing $y = Ax + B$ must be the tangent line to $y = \ln x$ at $x = 1$ The only way this will happen is if the function turns into its tangent line since any other line would generage a "sharp" change in direction and be non-differentiable.So let's find the tangent line to $y = \ln x$ at the point $(1, 0)$

$$y = \ln x \Longrightarrow y' = \frac{1}{x}$$

Hence, we'll need the slope to be equal to $y' = 1$. Hence the line must be $y = 0 + 1(x - 1)$, which becomes $y = x - 1$

Solution: **Marking scheme:** 1pt for using continuity; 1pt for setting the derivatives of each side equal to each other First we will need it to be continuous:

$$\begin{aligned} A + B &= \ln 1 \\ A &= -B \end{aligned}$$

Next we'll need the two derivative to line up:

$$A = \frac{1}{x} \overset{@x=1}{\Longrightarrow} A = 1 \Longrightarrow B = -1$$

Full-solution problems: Justify your answers and **show all your work**. Place a box around your final answer. Unless otherwise indicated, **simplification of answers is required in these questions**.

10 marks 5. Let $f(x) = \dfrac{x}{x^2+1}$. Use the definition of the derivative to find $f'(2)$. No marks will be given for the use of any differentiation rules.

Solution: **Marking scheme:** 2pt for limit def; 1pt for subbing in f(x) and f(2) properly; 2pt for getting a common denominator; 1 pt for simplifying; 2pt for factoring; 1pt for simplification; 1pt for final answer; Deductions: -1pt for missing limits; -1 pt for aesthetics .

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{x}{x^2+1} - \frac{2}{5}}{x-2} \\ &= \lim_{x\to 2} \frac{1}{x-2} \cdot \left[\frac{5x}{5(x^2+1)} - \frac{2(x^2+1)}{5(x^2+1)} \right] \\ &= \lim_{x\to 2} \frac{1}{x-2} \cdot \left[\frac{5x - 2x^2 - 2}{5(x^2+1)} \right] \\ &= \lim_{x\to 2} \frac{-(2x^2 - 5x + 2)}{5(x-2)(x^2+1)} \\ &= \lim_{x\to 2} \frac{-(x-2)(2x-1)}{5(x-2)(x^2+1)} \\ &= \lim_{x\to 2} \frac{-(2x-1)}{5(x^2+1)} \\ &= \boxed{\frac{-3}{25}} \end{aligned}$$

10 marks

6. A company has been selling 2000 television sets a week at \$420 each. Their weekly fixed costs are \$140,000 and the unit variable cost is \$140 per TV. A market survey indicates that for each \$18 rebate offered to a buyer, the number of sets sold will increase by 180 per week.

   (a) Find the linear demand equation. Use the notation p for the unit price and q for the ~~monthly~~ weekly demand.

   Answer: $q + 10p = 6200$

   Solution: Marking scheme: 1pt for correct slope, 1pt for correct point; 1pt for correct line. Full marks award for solving for either the weekly demand or monthly demand.
   A data point is (q, p) = (quantity, price). So the demand curve will be given by $p = mq + b$ and will have slope

   $$m = \frac{rise}{run} = \frac{\Delta p}{\Delta q} = \frac{-18}{180} = \frac{-1}{10}.$$

   So the equation becomes

   $$p = 420 - \frac{1}{10}(q - 2000) \Rightarrow p = 620 - \frac{1}{10}q$$

   (b) How should it set the size of the rebate (i.e. discount) to maximize its profit?

   Answer: \$40

   Solution: Marking scheme: 1pt for correct Revenue, Cost and Profit functions. These can be in the same line - 3 marks for it. 1pt for each of the 4 steps: differentiating or completing square to maximize; get the right q; get the right p; get the rebate. If one uses p instead of q as the independent variable, of course the 2nd step will be absent.
   We know that $R = \frac{-1}{10}q^2 + 620q$ and Proft will be given by:

   $$P = R - C = \frac{-1}{10}q^2 + 620q - (140000 + 140q) = \frac{-1}{10}q^2 + 480q - 140000$$

   To find its maximum we can take the midpoint of the roots, find the vertex by completing the square, or set its derivative to 0. We'll do the latter:.

   $$\frac{dP}{dq} = \frac{-2}{10}q + 480$$

   So that max profit will occur when $\frac{-2}{10}q + 480 = 0 \Rightarrow q = 2400$, from which we see that price must be $620 - 240 = 380$ Hence, the rebate must be $\boxed{\$40}$.

For students for found the month demand curve:

10 marks

7. A company has been selling 2000 television sets a week at \$420 each. Their weekly fixed costs are \$140,000 and the unit variable cost is \$140 per TV. A market survey indicates that for each \$18 rebate offered to a buyer, the number of sets sold will increase by 180 per week.

(a) Find the linear demand equation. Use the notation p for the unit price and q for the **monthly** demand.

Answer: $q + 40p = 24,800$

Solution: We need to convert some units first. Let $q =$ the number of TV's sold per month. Then our information is:

- 8000 sets per month at \$420 per TV
- Fixed Costs = \$560,000 per month unit variable cost is \$180 per TV
- An \$18 rebate corresponds to an increase of 720 sets per week.

A data point is $(q, p) =$ (quantity, price). So the demand curve will be given by $p = mq + b$ and will have slope

$$m = \frac{rise}{run} = \frac{\Delta p}{\Delta q} = \frac{-18}{4 \cdot 180} = \frac{-1}{40}.$$

So the equation becomes

$$p = 420 - \frac{1}{40}(q - 8000) \Rightarrow p = 620 - \frac{1}{40}q$$

(b) How should it set the size of the rebate (i.e. discount) to maximize its profit?

Answer: \$40

Solution: We know that $R = \frac{-1}{40}q^2 + 620q$ and Proft will be given by:

$$P = R - C = \frac{-1}{40}q^2 + 620q - (560000 + 140q) = \frac{-1}{40}q^2 + 480q - 560,000$$

To find its maximum we can take the midpoint of the roots, find the vertex by completing the square, or set its derivative to 0. We'll do the latter:.

$$\frac{dP}{dq} = \frac{-2}{40}q + 480$$

So that max profit will occur when $\frac{-2}{40}q + 480 = 0 \Rightarrow q = 9600$, from which we see that price must be $620 - \frac{1}{40} \cdot 9600 = 620 - 240 = 380$ Hence, the rebate must be $\boxed{\$40}$.