THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENTS OF APPLIED MATHEMATICS AND MATHEMATICS

Calculus 050a Midterm Examination

Friday, October 19, 2007 **Code 111** 7:00 p.m. -9:30 p.m.

INSTRUCTIONS

1. ALL QUESTIONS IN PART B MUST BE ANSWERED IN THE SPACE PROVIDED. Be sure to answer each part of a question in the space provided for that part of the question. INDICATE YOUR ANSWER CLEARLY.
2. DO NOT UNSTAPLE THE BOOKLET.
3. SHOW ALL YOUR WORK FOR PROBLEMS IN PART B. All results must be justified unless you are instructed otherwise. Unjustified answers will receive little or no credit.
4. Calculators and other aids are NOT allowed.
5. Questions start on Page 1 and continue to Page 17. Should you require extra space for any answer, Page 18 is provided for this purpose. Be sure that your booklet is complete.
6. TOTAL MARKS = 100.
7. Fill in the top of this page. Circle your section in the list below.

001	M. Khalkhali	007	P. Barmby
002	M. Tvalavadze	008	A. Buchel
003	T. Foth	009	W. Wu
004	C. Jones	010	M. Dawes
005	C. Jones	011	M. Khalkhali
006	N. Kiriushcheva	012	J. Malagon-Lopez
		570KC	R.N. Bryan

PART A (50 marks)

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 *marks* A1. Find the exponential function $f(x) = Ca^x$, where $a > 0$, whose graph passes through the points with coordinates $(0, -1)$ and $(2, -25)$.

A

$(0,-1)$ $(2,-25)$

so $f(0) = Ca^0 = -1 \Rightarrow C = -1$

so $f(x) = -a^x$

$f(2) = -a^2 = -25$

$a = \pm 5$ $(a > 0)$ so $a = 5$

$f(x) = -5^x$

A: $-(5^x)$	B: $(-5)^x$	C: 5^{-x}	D: $(-5)^{-x}$	E: $-\left(\frac{1}{5}\right)^x$

2 *marks* A2. Find a formula for the inverse function $f^{-1}(x)$ where $f(x) = \frac{1}{2}\ln(x+6)$.

C

$f(x) = \frac{1}{2} \ln(x+6)$

suppose $y = \frac{1}{2} \ln(x+6)$

$2y = \ln(x+6)$

$e^{2y} = (x+6)$

$e^{2y} - 6 = x$

$y = e^{2x} - 6$

A: $e^{2x} + 6$	B: $e^x - 6$	C: $e^{2x} - 6$	D: $\frac{1}{2}e^{x+6}$	E: $2e^{2x+6}$

2 marks A3. Find the exact value of $\log_3 6 - \log_3 18$.

B

$= \log_3 \frac{6}{18}$

$= \log_3 \frac{1}{3}$

$= \log_3 3^{-1}$

$= -1$

A: 1	B: -1	C: 2	D: -2	E: 3

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A4. Find the exact value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

E

suppose $\sin y = -\frac{\sqrt{3}}{2}$ $x \in [-1, 1]$

$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = -\frac{\pi}{3}$

A: $\frac{\pi}{3}$	B: $\frac{\pi}{6}$	C: $\frac{\pi}{2}$	D: $-\frac{\pi}{6}$	E: $-\frac{\pi}{3}$

2 marks A5. Find the exact value of $\cos\left[\sin^{-1}\left(\frac{5}{6}\right)\right]$.

D

suppose $\sin y = \frac{5}{6} = \frac{y}{r}$

$\cos y = \frac{x}{r}$ $x = \sqrt{r^2 - y^2} = \sqrt{36-25} = \sqrt{11}$

$\cos y = \frac{\sqrt{11}}{6}$

A: $\frac{5}{6}$	B: $\frac{\sqrt{11}}{5}$	C: $\frac{6}{5}$	D: $\frac{\sqrt{11}}{6}$	E: $\frac{6}{\sqrt{11}}$

2 marks A6. Find the exact value of $\arccos\left[\cos\left(\frac{5\pi}{4}\right)\right]$.

A

$\arccos\theta$

$\cos(\frac{5}{4}\pi) = \cos(\frac{\pi}{4})$

$x \in [-1, 1]$

$y \in [0, \pi]$

$= \cos(2\pi - \frac{3}{4}\pi)$

$= \cos\frac{3}{4}\pi$

A: $\frac{3\pi}{4}$	B: $\frac{5\pi}{4}$	C: $\frac{7\pi}{4}$	D: $-\frac{\pi}{4}$	E: $-\frac{3\pi}{4}$

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A7. Find the exact value of $\sin\left[2\arctan\left(\frac{1}{3}\right)\right]$.

D

suppose ~~y=~~ $\tan y = \frac{1}{3}$ $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\arctan\frac{1}{3} = m$

$\sin 2m = 2\sin m \cos m.$

$= 2\sin(\arctan\frac{1}{3})\cos(\arctan\frac{1}{3})$

$= 2 \times \frac{1}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{3}{5}$

A: $\frac{1}{\sqrt{10}}$	B: $\frac{3}{\sqrt{10}}$	C: $\frac{\sqrt{10}}{3}$	D: $\frac{3}{5}$	E: $\frac{2}{\sqrt{10}}$

2 marks A8. Determine $\lim_{x\to 2^+} \frac{7}{x-2}$.

D

$\because x \to 2^+ \quad \therefore x-2>0$

A: 7	B: -7	C: $-\infty$	D: ∞	E: 2

2 marks A9. Determine $\lim_{x\to-3} \frac{x+3}{|x+3|}$.

E.

$x \to -3^-$ so $|x+3| = -(x+3)$ $\lim_{x\to -3^-} \frac{x+3}{|x+3|} = \lim_{x\to -3^-} \frac{x+3}{-(x+3)} = -1$

$x \to -3^+$ so $|x+3| = (x+3)$ $\lim_{x\to -3^+} \frac{x+3}{|x+3|} = \lim_{x\to -3^+} \frac{x+3}{x+3} = 1$

$\lim_{x\to -3^-} \frac{x+3}{|x+3|} \neq \lim_{x\to -3^+} \frac{x+3}{|x+3|}$ DNE

A: 1	B: −1	C: 3	D: −3	E: does not exist

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A10. If, for all values of x between -1 and 1, $2 - \cos x \leq f(x) \leq 1 + \tan x$, find $\lim_{x \to 0} f(x)$.

~~B~~
D

$\lim_{x \to 0} 2 - \cos x = 2 - 1 = 1$

$x \in [-1, 1]$

$\lim_{x \to 0} 1 + \tan x = 1 + 0 = 1$

A: 2	B: 0	C: -1	D: 1	E: -2

2 marks A11. Determine $\lim_{x \to -\infty} \arctan(1 - x^2)$.

B

$\arctan(1 - \infty)$

$\arctan(-\infty)$

A: $\frac{\pi}{2}$	B: $-\frac{\pi}{2}$	C: ∞	D: $-\infty$	E: 0

2 marks A12. Determine $\lim_{x \to -\infty} \frac{\sqrt{4x^2+1}}{x+4}$.

E.E A

$$\lim_{x \to -\infty} \frac{\sqrt{4+\frac{1}{x^2}}}{1+\frac{4}{x}} = \frac{\sqrt{4+\frac{1}{\infty^2}}}{1+\frac{4}{(-\infty)}} = \frac{\sqrt{4+0}}{1-0} = \frac{2}{1}$$

A: 2	B: 1	C: ∞	D: $-\infty$	E: -2

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A13. Determine $\lim_{x\to 0^+} \ln(1-\cos x)$.

A

$= \lim_{x\to 0^+} \ln(1-\cos 0)$

$= \lim_{x\to 0^+} \ln 0$

$e^x = 0$

$x = \frac{1}{\infty}$

A: $-\infty$	B: ∞	C: 0	D: 1	E: e

2 marks A14. If $f(x) = \sin(e^x)$, find $f'(x)$.

B

$f'(x) = \sin(e^x)' = \cos e^x \cdot (e^x)' = e^x \cos e^x$

A: $e^x \sin(e^x)$	B: $e^x \cos(e^x)$	C: $-e^x \sin(e^x)$	D: $-e^x \cos(e^x)$	E: $e^{-x} \sin(e^x)$

2 marks A15. If $y = e^{\arctan x}$, find $\frac{dy}{dx}$.

C

$(\arctan x)' = \frac{1}{1+x^2}$

$y' = e^{\arctan x} \cdot (\arctan x)'$

$= \frac{e^{\arctan x}}{1+x^2}$

A: $e^{\arctan x} \sec^2 x$	B: $\frac{e^{\arctan x}}{\tan^2 x}$	C: $\frac{e^{\arctan x}}{1+x^2}$	D: $e^{\arctan x}$	E: $\frac{e^{\arctan x} \sec^2 x}{\tan^2 x}$

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A16. If $y = \arctan(\cos(x^2))$, find $\dfrac{dy}{dx}$.

C

$y' = \arctan'(\cos x^2) = \dfrac{1}{1+(\cos x^2)^2} \cdot (\cos x^2)'$

$= \dfrac{-\sin x^2 \cdot (x^2)'}{1+\cos^2 x^2}$

$= \dfrac{-2x \sin x^2}{1+\cos^2 x^2}$

A: $\dfrac{\sin(x^2)}{1+\cos(x^2)}$	B: $\dfrac{2x\sin(x^2)}{1+\cos^2(x^2)}$	C: $\dfrac{-2x\sin(x^2)}{1+\cos^2(x^2)}$
D: $\dfrac{2x\sin(x^2)}{\sqrt{1-\cos^2(x^2)}}$	E: $\sec[\cos(x^2)]\tan[\cos(x^2)]$	

2 marks A17. Suppose $F(x) = f(g(x))$ and $f(1) = -1, f(-2) = 3, f'(-2) = 6, g(1) = -2, g(-2) = 5$ and $g'(1) = 4$. Find $F'(1)$.

E

$F(x) = f(g(x))$

$F'(x) = f'(g(x)) \cdot g'(x)$

$F'(1) = f'(g(1)) \cdot g'(1)$

$= f'(-2) \cdot 4$

$= 6 \times 4$

$= 24$

A: -4	B: -8	C: -6	D: -12	E: 24

2 marks A18. Suppose $f(x) = 6^{\sec x}$. Find $f'\left(\frac{\pi}{4}\right)$.

D

$f(x)' = (6^{\sec x})' = 6^{\sec x} \cdot \ln 6 \cdot (\sec x)'$ $\qquad 6^{\sqrt{2}} \cdot \ln 6 \cdot \sqrt{2}$.

$= 6^{\sec x} \cdot \ln 6 \cdot \sec x \tan x$

$x = \frac{\pi}{4}$

$6^{\sec \frac{\pi}{4}} \cdot \ln 6 \cdot \sec \frac{\pi}{4} \tan \frac{\pi}{4}$

A: 36	B: $36\sqrt{2}$	C: $\left(6^{\sqrt{2}}\right)(\ln 6)$	D: $(\sqrt{2})\left(6^{\sqrt{2}}\right)\ln 6$	E: $\left(12^{\sqrt{2}}\right)(\ln 6)$

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

2 marks A19. Determine $\lim_{x\to 0} \frac{\sin(3x)}{x}$.

A

$$= \lim_{x\to 0} \frac{3\sin(3x)}{3x}$$

$$= \lim_{x\to 0} 3\times 1$$

$$= 3.$$

A: 3	B: $\frac{1}{3}$	C: -3	D: $-\frac{1}{3}$	E: does not exist

2 marks A20. Determine $\frac{d}{dx}(\ln 10)$.

C

$y = \ln 10$ constant.

$y' = (\ln 10)' = 0$

A: $\frac{1}{10}$	B: e^{10}	C: 0	D: 1	E: e

NOTE: YOUR ANSWERS TO THE PROBLEMS ON THIS PAGE MUST BE INDICATED ON THE SCANTRON SHEET.

A21 - A25. For each of the following, choose the letter which labels the graph below and transfer each of your answers to the scantron sheet.

2 marks A21. The graph of $y = -e^{-x}$ is answer on the scantron sheet. ~~B~~ D

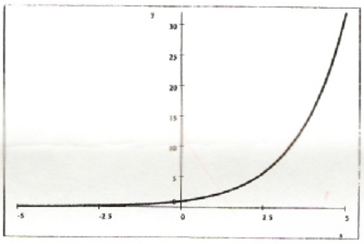
2 marks A22. The graph of $y = |\arctan x|$ is answer on the scantron sheet. C

2 marks A23. The graph of $y = \sin^{-1} x$ is answer on the scantron sheet. E.

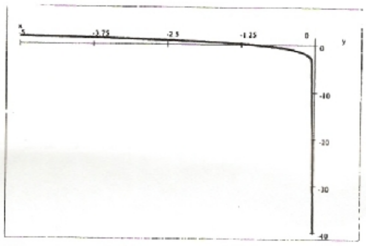
2 marks A24. The graph of $y = \ln(-x)$ is answer on the scantron sheet. ~~D~~ B

2 marks A25. The graph of $y = 2^x$ is answer on the scantron sheet. A

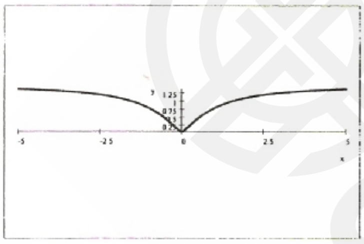
A



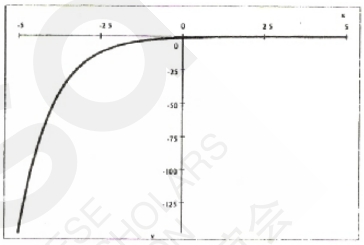
B



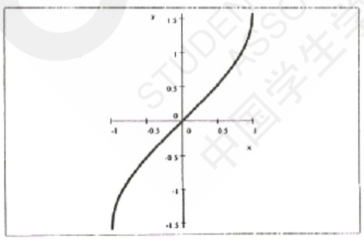
C



D



E



PART B (50 marks)

6 marks B26. Let $f(x)$ be the function given by

$$f(x) = \begin{cases} \ln x & \text{for } 1 \le x \\ 1 - x & \text{for } 0 \le x < 1 \\ e^x - 1 & \text{for } x < 0 \end{cases}$$

State the value of the indicated limit, if it exists, in the space provided. If a limit does not exist, write DNE.

(a) $\lim\limits_{x \to 0^-} f(x)$ 0

(b) $\lim\limits_{x \to 0^+} f(x)$ 1

(c) $\lim\limits_{x \to 0} f(x)$ DNE

(d) $\lim_{x\to 1^-} f(x)$ $\underline{\quad 0 \quad}$

6

(e) $\lim_{x\to 1^+} f(x)$ $\underline{\quad 0 \quad}$

(f) $\lim_{x\to 1} f(x)$ $\underline{\quad 0 \quad}$

4.5

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

~~5~~ marks B27. The function

$$f(x) = \begin{cases} cx - 1 & \text{if } x \le 1 \\ 2cx^2 + 3 & \text{if } x > 1 \end{cases}$$

is continuous at 1. What is the value of c? Justify your answer.

as function continuous at 1, =>? -0.5

$\lim_{x \to 1^-} cx - 1 = c - 1$ ①

$\lim_{x \to 1^+} 2cx^2 + 3 = 2c + 3$ ②

① = ②

so $c - 1 = 2c + 3$

$c = -1 - 3 = -4$

NOTE: SHOW ALL YOUR WORK FOR PART (ii) OF THE PROBLEM ON THIS PAGE.

5 marks B28. The limit $\lim_{h\to 0} \dfrac{\sec\left(\frac{\pi}{4}+h\right)-\sqrt{2}}{h}$ is the derivative of some function f at some number a.

(i) What are $f(x)$ and a? (Answers alone are sufficient.)

2

Answers: $f(x) =$ $\sec(\frac{\pi}{4}+h)$

$a =$ 0

(ii) Compute $\lim_{h\to 0} \dfrac{\sec\left(\frac{\pi}{4}+h\right)-\sqrt{2}}{h}$ by evaluating $f'(a)$.

$f'(x) = \sec'(\frac{\pi}{4}+h)$

$= \sec(\frac{\pi}{4}+h)\cdot\tan(\frac{\pi}{4}+h)$

$= \sec\frac{\pi}{4}\cdot\tan\frac{\pi}{4}$

$= \sqrt{2}$

3

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

6 marks B29. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation $2\cos x + \sin 2x = 0$.

$2\cos x + \sin 2x = 0$

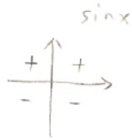
$2\cos x + 2\sin x \cos x = 0$

$2\cos x\,(1 + 2\sin x) = 0$

$2\cos x = 0$ or $1 + 2\sin x = 0$

$\cos x = 0$ $\quad$ $2\sin x = -1$

$\sin x = -\frac{1}{2}$



as $x \in [0, 2\pi]$

So for $\cos x = 0$, $\quad x = \frac{\pi}{2}, \frac{3}{2}\pi$

for $\sin x = -\frac{1}{2}$, $\quad x = \frac{7}{6}\pi, \frac{11}{6}\pi$

So $x = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$.

5

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

6 marks B30. Find an equation of the tangent line to the graph of $e^{y-2\pi}\cos x = 1 + \sin(xy)$ at the point $(0, 2\pi)$.

Suppose the equation of the tangent line is $y = kx + b$, slope k.

it through point $(0, 2\pi)$

so $b = 2\pi$. $y = kx + 2\pi$

also, $(e^{y-2\pi}\cos x)' = [1 + \sin(xy)]'$

$e^{y-2\pi}\cdot(y-2\pi)'\cdot\cos x + e^{y-2\pi}(-\sin x) = \cos xy \cdot y'$

$(\cos xy - e^{y-2\pi}\cos x)\,y' = -e^{y-2\pi}\sin x$

$\cancel{y' = -\dfrac{e^{y-2\pi}\sin x}{\cos xy - e^{y-2\pi}\cos x} = -\dfrac{e^{2\pi-2\pi}\sin 0}{\cos(0\cdot 2\pi) - e^{2\pi-2\pi}\cos 0} = -\dfrac{0}{1}} = 0$

at $(0, 2\pi)$ $e^{2\pi-2\pi}\cos 0 = 1 + \sin(0\cdot 2\pi) = 1$

so $k = 0$.

~~tangent line L $y = 2\pi$.~~

$$y' = \frac{e^{y-2\pi}\sin x}{e^{y-2\pi}\cos x + \cos xy} = k$$

$$\frac{e^{2\pi-2\pi}\sin 0}{e^{2\pi-2\pi}\cos 0 + \cos(0\cdot 2\pi)} = k$$

So tangent Line L is $y = 2\pi$

2

5 **NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.**

6 marks B31. Use the Intermediate Value Theorem to show that there is a root of (i.e., a solution of) the equation $\sin 3x + \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$.

Continuity (-1)

Suppose $f(x) = \sin 3x + \cos x$

$f(0) = \sin(3\times 0) + \cos(0)$
$= \sin 0 + \cos 0$
$= 0 + 1$
$= 1$

$f(\frac{\pi}{2}) = \sin(3\times\frac{\pi}{2}) + \cos(\frac{\pi}{2})$
$= \sin(\frac{3}{2}\pi) + \cos\frac{\pi}{2}$
$= -1 + 0$
$= -1$

use IVT. $f(\frac{\pi}{2}) \le f(x) \le f(0)$.

$-1 \le f(x) \le 1$

So $f(x)$ can equals to 0.

there is a root c that makes

$f(x) = \sin 3c + \cos c = 0$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

5 *marks* B32. Find $\frac{dy}{dx}$ if $y = \tan^2(1+e^{2x})$. Do not simplify your answer.

$y' = [\tan^2(1+e^{2x})]'$

way ① Suppose $a = \tan(1+e^{2x})$

$b = (1+e^{2x})$

$c = 2x$

5

$y' = (a^2)'$

$= 2a \cdot a' \cdot b' \cdot c'$

$= 2\tan(1+e^{2x}) \cdot \sec^2(1+e^{2x}) \cdot e^{2x} \cdot 2$

$= 2e^{2x}\, 2\tan(1+e^{2x}) \cdot \sec^2(1+e^{2x})$

way ② $y' = 2\tan(1+e^{2x}) \cdot [\tan(1+e^{2x})]'$

$= 2\tan(1+e^{2x}) \cdot \sec^2(1+e^{2x}) \cdot (1+e^{2x})'$

$= 2\tan(1+e^{2x}) \cdot \sec^2(1+e^{2x}) \cdot e^{2x} \cdot (2x)'$

$= 2\tan(1+e^{2x}) \cdot \sec^2(1+e^{2x}) \cdot 2e^{2x}$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEMS ON THIS PAGE.

5 *marks* B33. Find $\frac{dy}{dx}$ if $y = \arcsin(\sqrt[3]{\cos(x^2)})$. Do not simplify your answer.

suppose $a = x^2$

$b = \cos a = \cos x^2$

$c = \sqrt[3]{b} = \sqrt[3]{\cos(x^2)}$

$$y' = (\arcsin c)'$$

$$= \frac{1}{\sqrt{1-c^2}} \cdot c'$$

$$= \frac{1}{\sqrt{1-c^2}} \cdot \frac{1}{3} b^{-\frac{2}{3}} \cdot b'$$

$$= \frac{1}{\sqrt{1-c^2}} \cdot \frac{b^{-\frac{2}{3}}}{3} \cdot \cos' a$$

$$= \frac{1}{\sqrt{1-c^2}} \cdot \frac{b^{-\frac{2}{3}}}{3} \cdot (-\sin a) \cdot a'$$

$$= \frac{1}{\sqrt{1-(\sqrt[3]{\cos(x^2)})^2}} \cdot \frac{1}{3}[\cos(x^2)]^{(-\frac{2}{3})} \cdot (-\sin x^2) \cdot 2x$$

5/5

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

6 marks B34. Determine $\lim_{x\to\infty} \frac{1}{\arctan\left[\ln\left(\frac{1}{x}\right)\right]}$.

6

$$\lim_{x\to+\infty} \frac{1}{\arctan[\ln(\frac{1}{x})]}$$

$$= \frac{1}{\arctan[\ln 0]}$$

$$= \frac{1}{\arctan(-\infty)}$$

$$= \frac{1}{(-\frac{\pi}{2})}$$

$$= -\frac{2}{\pi}$$

$\arctan x$

$x \in R$

$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\lim_{x\to-\infty} \frac{1}{\arctan[\ln(\frac{1}{x})]}$$

$$= \frac{1}{\arctan[\ln\frac{1}{(-\infty)}]}$$

$$= \frac{1}{\arctan(\ln 0)}$$

$$= \frac{1}{\arctan(-\infty)}$$

$$= -\frac{2}{\pi}$$

as $\lim_{x\to+\infty} \frac{1}{\arctan[\ln(\frac{1}{x})]} = \lim_{x\to-\infty} \frac{1}{\arctan[\ln(\frac{1}{x})]}$

So $\lim_{x\to\infty} \frac{1}{\arctan[\ln(\frac{1}{x})]} = -\frac{2}{\pi}$