MAT 1332, Winter 2016, Assignment 6

Due Monday April 11 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Petko Kitanov Catalin Rada

Student Name ____________________________ Student Number ____________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

QUESTION 1. Find the (2,2)-entry of the Jacobian matrix of the function

$$F(x,y) = \begin{bmatrix} x^2e^y + 2x\sin(x^y) \\ x^4\sin(\ln(x^2)) - 3ye^{-2x} \end{bmatrix}$$

at the point $(2,1)$.

The Jacobian of $F = \begin{pmatrix} f \\ g \end{pmatrix}$ is given by

$$J = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}.$$

The (2,2) entry means the second row, second column ($(i,j)^{\text{th}}$ entry is the entry in the i^{th} row and j^{th} column); that is, $\frac{dg}{dy}$.

Here g, the second row of F, is the function $x^4\sin(\ln(x^2)) - 3ye^{-x}$. Its y-partial derivative is

$$\frac{dg}{dy} = \frac{d}{dy}\left[x^4\sin(\ln(x^2)) - 3ye^{-2x}\right] = 0 - 3e^{-2x}.$$

We are asked to evaluate this at the point $(2,1)$; in other words, plug in 2 for x and 1 for y (although there isn't a y value in this case):

$$\frac{dg}{dy}(2,1) = -3e^{-2(2)} = -3e^{-4}.$$

QUESTION 2. Consider the following system of linear differential equations:

$$\frac{dx}{dt} = -3x + 4y$$
$$\frac{dy}{dt} = x - 3y$$

(a) Find the eigenvalues and eigenvectors associated with the system.

The matrix associated to the system is the matrix of coefficients in the differential equations:

$$A = \begin{pmatrix} -3 & 4 \\ 1 & -3 \end{pmatrix}.$$

Its eigenvalues are the roots of

$$\det(A - \lambda I) = \det \begin{pmatrix} -3-\lambda & 4 \\ 1 & -3-\lambda \end{pmatrix} = (-3-\lambda)(-3-\lambda) - 4 = \lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1),$$

so $\lambda = -5, -1$. (We could use the quadratic formula instead of factoring directly; in general the quadratic formula will be necessary.)

We find the eigenvectors for each eigenvalue λ by solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ by row reduction. For $\lambda = -5$,

$$A - \lambda I = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix},$$

and the augmented matrix is row-reduced to

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The general solution to this (i.e., to $x_1 + 2x_2 = 0$) is

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2r \\ r \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} r, r \in \mathbb{R}, r \neq 0.$$

Similarly we find that

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} s, s \in \mathbb{R}, s \neq 0$$

are the eigenvectors for the eigenvalue $\lambda_2 = -1$.

(b) Write down the general solution formula for the system.

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2C_1 e^{-5t} + 2C_2 e^{-t} \\ C_1 e^{-5t} + C_2 e^{-t} \end{pmatrix},$$

so $x(t) = -2C_1 e^{-5t} + 2C_2 e^{-t}$, $y(t) = C_1 e^{-5t} + C_2 e^{-t}$, with C_1 and C_2 arbitrary constants.

(c) Find $y(t)$ if $x(0) = 18, y(0) = 5$.

At $t = 0$, $x = -2C_1 + 2C_2$ and $y = C_1 + C_2$, so we need to solve the linear system

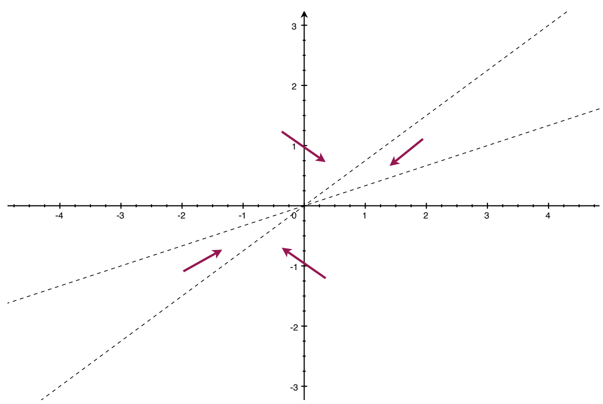
$$-2C_1 + 2C_2 = 18$$
$$C_1 + C_2 = 5$$

We can use any method we like, such as row-reduction, to find that there is a single solution, with $C_1 = -2$, $C_2 = 7$. The solution for $y(t)$ is therefore

$$y(t) = -2e^{-5t} + 7e^{-t}$$

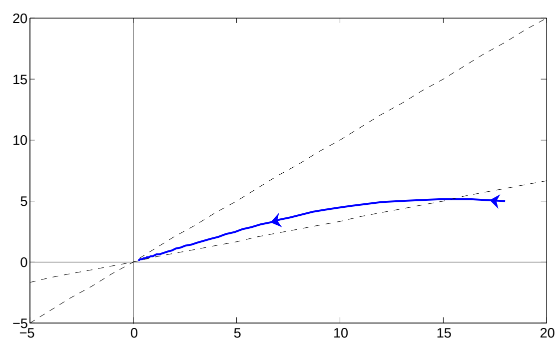
(d) Draw the x- and y-nullclines. Sketch a direction arrow in each quadrant.

The x-nullcline is the solutions to $-3x + 4y = 0$, which is the line $y = \frac{3}{4}x$. The y-nullcline is the solutions to $x - 3y = 0$, which is the line $y = \frac{1}{3}x$.



Direction arrows are obtained by taking various points (x, y) between the nullclines and plotting the vector $(-3x + y, 4x - 3y)$ originating from that point. Note that vectors are not drawn to scale.

(e) Carefully sketch the solution curve for the initial condition in part (c) in the phase plane (on a separate graph).



[2]

Note that the initial point $(18, 5)$ starts underneath the y-nullcline, so the trajectory is initially increasing. It is horizontal when it crosses the y-nullcline and then decreases to zero (the equilibrium) thereafter.

(f) Is the point $(0, 0)$ stable or unstable? Classify this equilibrium.

Note that $(0, 0)$ is the only equilibrium point. This will be true for any **linear** system of differential equations. Since both of the eigenvalues (-5 and -1) have negative real part, this equilibrium is stable. This equilibrium is a **sink**.

We can further see that the system is stable since the particular solution in part (c) heads towards $(0, 0)$ as time t gets larger and larger.

QUESTION 3. Consider the nonlinear system

$$x' = 12 - 0.1xy - 0.3x$$
$$y' = 0.1xy - 2y$$

in the region $x, y \geq 0$

(a) Draw the x- and y-nullclines.

The y-nullcline occurs when $y' = 0$, so

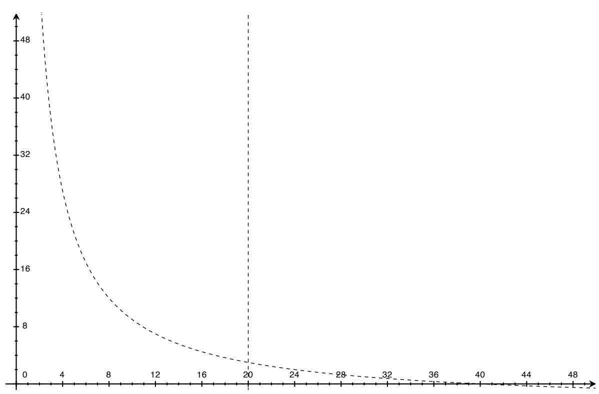
$$y(0.1x - 2) = 0$$

Thus $y = 0$ (the x-axis) or $x = 20$ (a vertical line).

The x-nullcline occurs when $x' = 0$, so

$$0.1xy = 12 - 0.3x$$
$$y = \frac{120}{x} - 3,$$

which is a hyperbola.



(b) Identify all equilibria.

Equilibria occur when the nullclines are equal. From the y-nullcline, we have $y = 0$ or $x = 20$. Substituting $x = 20$ into the x-nullcline, we find that $y = 3$. Substituting $y = 0$ into the x-nullcline, we find that $x = 40$. Thus the equilibria are

$$(40, 0) \text{ and } (20, 3).$$

(c) Sketch the phase portrait.

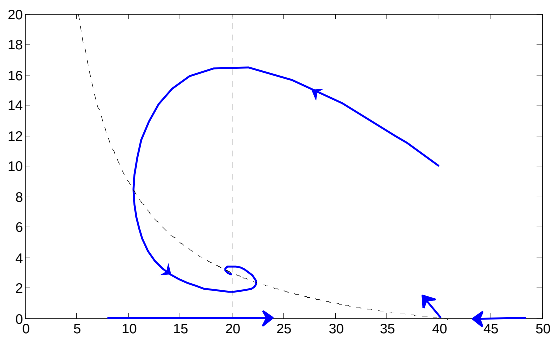
[2] We could find eigenvalues and look at the sign of the real part for each equilibria (which is fine) or we could determine the direction in each quadrant. In fact, knowing the nullclines, we can probably get away with testing just a single point (though you can test more if you want).

Testing (say) the point $(40, 10)$, we have

$$x' = 12 - 0.1(40)(10) - 0.3(40) = -40 < 0$$
$$y' = 0.1(40)(10) - 2(10) = 20 > 0$$

The numbers aren't important, just the signs. So that means solutions are facing north-west in the upper right quadrant.

Along the x-axis, solutions are always horizontal (because of the y-nullcline). Below $x = 40$, they are increasing. Above $x = 40$, they are decreasing.



QUESTION 4. Consider a disease that propagates according to the system

$$\frac{dx}{dt} = 16 - 0.2xy - 0.4x$$
$$\frac{dy}{dt} = 0.1xy - 8y$$

where x represents susceptible individuals, y represents infected individuals.

(a) Find all biologically meaningful steady states.

Biologically meaningful here simply means that the numbers are not negative. The steady states (= equilibrium points) are the places where both $16 - 0.2xy - 0.4x = 0$ and $0.1xy - 8y = 0$. The second equation is easier (since we can factor it) so we deal with it first: $y(0.1x - 8) = 0$ when $y = 0$ or when $x = 8/0.1 = 80$. For each of these cases we plug the given value into the first equation (which must also hold).

[1] If $y = 0$, then the first equation says that $16 - 0.4x = 0$, so $x = 16/0.4 = 40$. Therefore $(40, 0)$ is one equilibrium.

[1] The only other case is when $x = 80$. Here, the first equation says that $16 - 0.2(80)y - 0.4(80) = 0$, so $16y = -16$ and $y = -1$. Therefore $(60, -1)$ is another equilibrium, and there are no others. This equilibrium point is not biologically meaningful since its second coordinate is negative. **(1 point for ruling out this equilibrium with an explanation)**

(b) Show that the Jacobian matrix of this system is given by

$$\begin{bmatrix} -0.4 - 0.2y & -0.2x \\ 0.1y & 0.1x - 8 \end{bmatrix}$$

The Jacobian of

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 16 - 0.2xy - 0.4x \\ 0.1xy - 8y \end{pmatrix}$$

is given by

$$J = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}.$$

We just have to confirm four partial derivatives were given correctly. So, for example,

$$\frac{d}{dx}(16 - 0.2xy - 0.4x) = -0.4 - 0.2y.$$

(c) For the biologically meaningful steady states from (a), find the eigenvalues of the Jacobian matrix.

[2] We have one biologically meaningful steady state: (40,0). We plug $x = 40$, $y = 0$ into the formula given in part (b) for J:

$$J(40, 0) = \begin{pmatrix} -0.4 & -8 \\ 0 & -4 \end{pmatrix}.$$

This matrix is upper-triangular (since the only entry below the main diagonal is zero), so its eigenvalues are its diagonal entries: -0.4 and -4. You should compute the characteristic polynomial to verify that these are in fact the eigenvalues.

(d) Determine the stability of the biologically meaningful steady states.

[1] Since the eigenvalues of the Jacobian matrix at the equilibrium have negative real part (in fact, are negative real numbers), we can conclude that this equilibrium is stable. In fact, it is a stable sink.

What this means in concrete terms is that starting from any population with any infection rate, after enough time the end result will be that x is very close to 40 and y is very close to 0; in particular the disease will be wiped out in time.