

MAT 1332, Winter 2016, Assignment 5

Due Wednesday March 23 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name \_\_\_\_\_ Student Number \_\_\_\_\_

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature \_\_\_\_\_

QUESTION 1. Determine all values of  $x$  that satisfy  $\det(A) = 0$ , where

$$A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$$

[3]

We have

$$\det(A) = 1 + 2x^3 - 3x^2 = 1 - x^2 + 2x^3 - 2x^2.$$

Factoring gives us

$$\det(A) = (1 - x)(1 + x) + 2x^2(x - 1) = (1 - x)(1 + x - 2x^2) = (1 - x)(1 - x)(1 + 2x)$$

so  $\det(A) = 0$  implies  $x = 1$  or  $x = -1/2$ .

QUESTION 2. Suppose  $A$ ,  $B$  and  $C$  are invertible  $4 \times 4$  dimensional matrices with the properties that  $\det(A) = 3$ ,  $\det(B) = 5$  and  $\det(C) = 2$ . Calculate the determinant of

$$(3A^{-1}BCC^T A^3).$$

Using properties of determinants, we have

$$\det(3A^{-1}BCC^T A^3) = 3^4 \times \det(A^{-1}) \times \det(B) \times \det(C) \times \det(C^T) \times \det(A^3)$$

Hence

$$3^4 \frac{1}{\det(A)} \times \det(B) \times \det(C)^2 \times \det(A)^3 = \frac{3^4}{3} \times 5 \times 2^2 \times 3^3 = 14580$$

QUESTION 3. Consider the system of equations

$$\begin{aligned} y_1' &= 10y_1 - 5y_2 \\ y_2' &= 8y_1 - 12y_2 \end{aligned}$$

a) Find the general solution.

[4]

We have

$$\det(A - \lambda I) = \lambda^2 + 2\lambda - 80 = (\lambda - 8)(\lambda + 10)$$

so the eigenvalues are  $\lambda_1 = 8$  and  $\lambda_2 = -10$ .

Corresponding to  $\lambda_1 = 8$ , we have

$$\left[ \begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

It follows that the eigenvectors are in the form

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} r, r \in \mathbb{R}, r \neq 0$$

Corresponding to  $\lambda_2 = -10$ , we have

$$\left[ \begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 4 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

so eigenvectors are in the form

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} s, s \in \mathbb{R}, s \neq 0$$

Hence the general solution is

$$\vec{Y}(t) = c_1 e^{8t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + c_2 e^{-10t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

or  $y_1(t) = 5c_1 e^{8t} + c_2 e^{-10t}$  and  $y_2(t) = 2c_1 e^{8t} + 4c_2 e^{-10t}$ .

b) Find the solution satisfying  $y_1(0) = 7$  and  $y_2(0) = -8$ .

[2]

We have  $y_1(0) = 5c_1 + c_2 = 7$  and  $y_2(0) = 2c_1 + 4c_2 = -8$ . Solving, we have  $c_1 = 2$  and  $c_2 = -3$ . Hence the particular solution is

$$\vec{Y}(t) = 2e^{8t} \begin{pmatrix} 5 \\ 2 \end{pmatrix} - 3e^{-10t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

QUESTION 4. Find the general solution of the system

$$\begin{aligned}x' &= -x + 5y + 4z \\y' &= -x + 5y - z \\z' &= -6z\end{aligned}$$

[6]

We have

$$\det(A - \lambda I) = -\lambda(\lambda + 6)(\lambda - 4) = 0$$

so the eigenvalues are  $\lambda_1 = 0$  and  $\lambda_2 = -6$  and  $\lambda = 4$ .

Corresponding to  $\lambda_1 = 0$ , we have

$$\left[ \begin{array}{ccc|c} -1 & 5 & 4 & 0 \\ -1 & 5 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} -1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

It follows that the eigenvectors are

$$\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} r, r \in \mathbb{R}, r \neq 0.$$

Corresponding to  $\lambda_2 = -6$ , we have

$$\left[ \begin{array}{ccc|c} 5 & 5 & 4 & 0 \\ -1 & 11 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 49/60 & 0 \\ 0 & 1 & -1/60 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

so the eigenvectors are

$$\begin{pmatrix} -49 \\ 1 \\ 60 \end{pmatrix} s, s \in \mathbb{R}, s \neq 0$$

Corresponding to  $\lambda_3 = 4$ , we have

$$\left[ \begin{array}{ccc|c} -5 & 5 & 4 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & -10 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

so the eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} w, w \in \mathbb{R}, w \neq 0$$

It follows that the general solution is

$$\vec{X}(t) = c_1 \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -49 \\ 1 \\ 60 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$