MAT 1332, Winter 2016, Assignment 3
Due Wednesday February 10 in the math department dropboxes by 7:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Petko Kitanov Catalin Rada

DGD (circle one): 1 2 3 4

Student Name ____________________________ Student Number ____________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

Question 1. Consider the differential equation

$$\frac{dP}{dt} = P^3 - 5P^2 + 6P$$

a) Find all equilibrium points.

b) Determine the stability of each equilibrium point.

c) Draw a phase-line diagram.

[4] (a) Factoring, we have

$$P^3 - 5P^2 + 6P = P(P^2 - 5P + 6) = P(P-3)(P-2) = F(P)$$

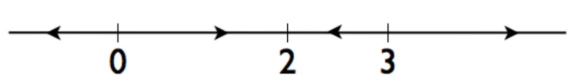
Hence the equilibria are $P_1 = 0; P_2 = 2$ and $P_3 = 3$.
(b) Differentiating, we have

$$F'(P) = 3P^2 - 10P + 6.$$

Substituting the equilibrium points, we have

$$F'(0) = 6 > 0 \text{ so this point is unstable}$$
$$F'(2) = -2 < 0 \text{ so this point is stable}$$
$$F'(3) = 3 > 0 \text{ so this point is unstable}$$

(c) The phase-line diagram is thus:



QUESTION 2. A bacterial culture starts with 600 bacteria and expands at a rate proportional to its amount. After 3 hours, there are 1000 bacteria.

a) Determine the expression for the number of bacteria as a function of time.

b) How many bacteria are there after 8 hours?

c) How long does it take until there are 40,000 bacteria?

(a) We have $\frac{dP}{dt} = KP$, so separating variables and integrating gives

$$\begin{aligned} \frac{dP}{P} &= K dt \\ \ln P &= Kt + C \\ P &= Ae^{Kt} \end{aligned}$$

Applying the conditions $P(0) = 600$ and $P(3) = 1000$ gives

$$\begin{aligned} 600 &= Ae^0 & A &= 600 \\ 1000 &= 600e^{3K} \\ e^{3K} &= \frac{5}{3} \\ 3K &= \ln\frac{5}{3} \\ K &= \frac{1}{3}\ln\frac{5}{3} = 0.170275 \end{aligned}$$

Hence the solution is $P(t) = 600e^{0.170275t}$

(b) After 8 hours there are $P(8) = 600e^{0.170275\times 8} = 2342.86 \approx 2342$ bacteria. (You can't have a fraction of a bacterium.)

(c) We have

$$\begin{aligned} 40000 &= 600e^{0.170275\bar{t}} \\ \bar{t} &= \frac{\ln(200/3)}{0.170275} = 24.66 \text{ hours} \end{aligned}$$

Question 3. Uranium U-235 has a half-life of approximately 7×10^8 years. The decay rate of uranium is proportional to the quantity present. After a nuclear accident, 100 grams of uranium is released into the air.

a) Write down the differential equation describing this system.

b) Find the function describing the amount of uranium remaining after time t, where t is time in years since the nuclear accident.

c) How much uranium is there after two million years? How much is there after two billion years?

[4] (a) Since the uranium decays, the differential equation is negative. Hence we have $\frac{dQ}{dt} = -Kt$, where $Q(t)$ is the quantity of uranium remaining at time t.

(b) Separating variables and integrating (just as in Question 1), we have

$$Q = Ae^{-kt}$$

The initial condition gives $Q(0) = A = 100$, while the halflife condition says that $Q(7 \times 10^8) = 100e^{-700000000k} = 50$, so it follows that

$$k = -\frac{1}{700000000} \ln \frac{1}{2} = -\frac{\ln 2}{700000000}.$$

Hence the solution is

$$Q(t) = 100 \exp\left(-\frac{\ln 2}{700000000} t\right)$$

(c) After two million years, we have

$$\begin{aligned} Q(2000000) &= 100 \exp\left(-\frac{\ln 2}{700000000} 2000000\right) \\ &= 100 \exp\left(-\frac{\ln 2}{700} 2\right) \\ &= 99.8 \text{ grams} \end{aligned}$$

After two billion years, we have

$$\begin{aligned} Q(2000000000) &= 100 \exp\left(-\frac{\ln 2}{700000000} 2000000000\right) \\ &= 100 \exp\left(-\frac{\ln 2}{700} 2000\right) \\ &= 13.8 \text{ grams} \end{aligned}$$

QUESTION 4. Solve the following differential equations:

a) $\dfrac{dy}{dx} = \dfrac{e^{7x-2}}{e^{3x+4y}}$

Separating variables and integrating, we have

$$\begin{aligned} e^{4y} dy &= \frac{e^{7x-2}}{e^{3x}} dx \\ \int e^{4y} dy &= \int \frac{e^{7x-2}}{e^{3x}} dx \\ &= \int e^{4x-2} dx \\ \frac{e^{4y}}{4} &= \frac{e^{4x-2}}{4} + c \\ e^{4y} &= e^{4x-2} + K \\ y &= \frac{1}{4} \ln\left(e^{4x-2} + K\right) \end{aligned}$$

(Note that the constant cannot be moved outside the logarithm.)

b) $y^3 \dfrac{dy}{dx} = 5x^4 y^4 - 40x^4$

[2] Factoring the right-hand side, then separating variables and integrating, we have

$$\begin{aligned} y^3 \frac{dy}{dx} &= 5x^4(y^4 - 8) \\ \frac{y^3}{y^4 - 8} dy &= 5x^4 dx \\ \frac{1}{4} \int \frac{4y^3}{y^4 - 8} dy &= \int 5x^4 dx \\ \frac{1}{4} \ln|y^4 - 8| &= x^5 + c \\ \ln|y^4 - 8| &= 4x^5 + K \\ y^4 - 8 &= Ae^{4x^5} \\ y^4 &= Ae^{4x^5} + 8 \\ y &= \sqrt[4]{Ae^{4x^5} + 8} \end{aligned}$$

c) $y' = y\cot(\pi t)$. Find the general solution and also the particular solution satisfying $y\left(\frac{1}{2}\right) = 3$.

[3] First rewrite the differential equation as $\frac{dy}{dt} = y\frac{\cos(\pi t)}{\sin(\pi t)}$. Then, separating variables and integrating, we have

$$\begin{aligned} \frac{1}{y}dy &= \frac{\cos(\pi t)}{\sin(\pi t)}dt \\ \int \frac{1}{y}dy &= \int \frac{\cos(\pi t)}{\sin(\pi t)}dt \end{aligned}$$

Let $u = \sin(\pi t)$. Then $\frac{du}{dt} = \pi\cos(\pi t)$ so $dt = \frac{du}{\pi\cos(\pi t)}$. We thus have

$$\begin{aligned} \int \frac{1}{y}dy &= \int \frac{\cos(\pi t)}{u}\frac{du}{\pi\cos(\pi t)} \\ &= \frac{1}{\pi}\int \frac{du}{u} \\ \ln|y| &= \frac{1}{\pi}\ln|u| + c \\ \ln|y| &= \frac{1}{\pi}\ln|\sin(\pi t)| + c \\ &= \ln|\sin(\pi t)|^{1/\pi} + c \\ y &= A|\sin(\pi t)|^{1/\pi} \end{aligned}$$

is the general solution.

The particular solution satisfying $y(0) = 3$ satisfies

$$\begin{aligned} 3 &= A|\sin(0)|^{1/\pi} && \text{so } A = 3 \\ y &= 3|\sin(\pi t)|^{1/\pi} \end{aligned}$$