

MAT 1332, Winter 2016, Assignment 2

Due Wednesday February 3 in the math department dropboxes by 7:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name \_\_\_\_\_ Student Number \_\_\_\_\_

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature \_\_\_\_\_

QUESTION 1. For each of the following integrals do the following:

(i) Explain in one sentence why the integral is an improper integral.

(ii) Decide whether the integral converges or diverges. Justify your answer.

$$(a) \int_0^4 \frac{e^{-1/t}}{t^2} dt \quad (b) \int_{-1}^1 \frac{8x^3 - 12x^2 + 6x - 1}{(2x - 1)^2} dx \quad (c) \int_2^5 \ln(5 - x) dx.$$

(i) (a)  $\int_0^4 \frac{e^{1/t}}{t^2} dx$  is an improper integral since  $f(t) = \frac{e^{1/t}}{t^2}$  is continuous on  $(0, 4]$  but not continuous on  $[0, 4]$ .

[1] (i) (b)  $\int_{-1}^1 \frac{8x^3 - 12x^2 + 6x - 1}{(2x - 1)^2} dx$  is an improper integral since  $g(x) = \frac{8x^3 - 12x^2 - 4x + 4}{(2x - 1)^2}$  has a vertical asymptote at  $x = \frac{1}{2}$ , which is contained in the interval  $[-1, 1]$ . [Note that even though the function ultimately simplifies, it's still an improper integral.]

(1) (c)  $\int_2^5 \ln(5 - x) dx$  is an improper integral since  $h(x) = \ln(5 - x)$  is continuous on  $[2, 5)$  but not continuous on  $[2, 5]$ .

(ii) (a)

Let  $u = -\frac{1}{t}$ , so  $\frac{du}{dt} = \frac{1}{t^2}$  and hence  $dt = t^2 du$ . Then we have

$$\begin{aligned}\int_0^4 \frac{e^{-1/t}}{t^2} dx &= \int_{t=0}^{t=4} \frac{e^u}{t^2} t^2 du \\ &= \int_{t=0}^{t=4} e^u du \\ &= e^u \Big|_{t=0}^{t=4} \\ &= e^{-1/t} \Big|_0^4 \\ &= e^{-1/4} - \lim_{\epsilon \rightarrow 0^+} e^{-1/\epsilon} \\ &= e^{-1/4} - 0\end{aligned}$$

It follows that this integral converges.

(ii) (b)

This clearly isn't an integration by parts question (as it's not a product) and it's not really a substitution question either (as the obvious substitutions have derivatives that don't cancel). It's not a partial fractions question either, since the denominator is already in its basic form. This means the only remaining choice is to manipulate the function.

Either by factoring or long division, we can rewrite the integral as

$$\begin{aligned}\int_{-1}^1 \frac{8x^3 - 12x^2 + 6x - 1}{(2x - 1)^2} dx &= \int_{-1}^1 \frac{(2x - 1)^3}{(2x - 1)^2} dx \\ &= \int_{-1}^1 (2x - 1) dx \\ &= [x^2 - x]_{-1}^1 \\ &= (1 - 1) - (1 - (-1)) \\ &= -2\end{aligned}$$

(Note that the discontinuity becomes irrelevant after simplifying, so we don't need to split the integral limits at  $\frac{1}{2}$ , although it's fine if you do of course.)

(ii) (c)

[3]

Using integration by parts, we have

$$\begin{aligned}u &= \ln(5 - x) & v' &= 1 \\ u' &= -\frac{1}{5 - x} & v &= x\end{aligned}$$

Then the integral is

$$\begin{aligned}
 \lim_{T \rightarrow 5^-} \left[ x \ln(5-x) \Big|_2^T + \int_2^T \frac{x}{5-x} dx \right] &= \lim_{T \rightarrow 5^-} \left[ x \ln(5-x) \Big|_2^T + \int_2^T \frac{x-5+5}{5-x} dx \right] \\
 &= \lim_{T \rightarrow 5^-} \left[ x \ln(5-x) \Big|_2^T + \int_2^T \left( -1 + \frac{5}{5-x} \right) dx \right] \\
 &= \lim_{T \rightarrow 5^-} \left[ x \ln(5-x) - x - 5 \ln(5-x) \Big|_2^T \right] \\
 &= \lim_{T \rightarrow 5^-} [(T-5) \ln(5-T) - T] - [-3 \ln 3 - 2] \\
 &= \lim_{T \rightarrow 5^-} \left[ \frac{\ln(5-T)}{\frac{1}{T-5}} - T \right] - [-3 \ln 3 - 2] \\
 &\stackrel{L'H}{=} \lim_{T \rightarrow 5^-} \left[ \frac{\frac{-1}{5-T}}{\left(\frac{1}{T-5}\right)^2} - T \right] - [-3 \ln 3 - 2] \\
 &= \lim_{T \rightarrow 5^-} [(T-5) - T] - [-3 \ln 3 - 2] \\
 &= -5 + 3 \ln 3 + 2 \\
 &= 3 \ln 3 - 3
 \end{aligned}$$

Hence the integral converges.

QUESTION 2. Evaluate the following indefinite integrals:

$$(a) \int \frac{2x^4 - 26x^3 + 72x^2 - 4x + 1}{x^2 - 13x + 36} dx \quad (b) \int \frac{x^3 - 16x^2 + 68x - 27}{x^2 - 16x + 64} dx \quad (c) \int \frac{2x^2 - x - 6}{(x^2 + 4)(x - 1)} dx$$

(a)

[4]

Since the degree of the numerator is larger than the degree of the denominator, we need to do long division:

$$\begin{array}{r}
 2x^2 \\
 x^2 - 13x + 36 \overline{) 2x^4 - 26x^3 + 72x^2 - 4x + 1} \\
 \underline{2x^4 - 26x^3 + 72x^2} \phantom{- 4x + 1} \\
 -4x + 1
 \end{array}$$

Hence  $2x^4 - 26x^3 + 72x^2 - 4x + 1 = 2x^2(x^2 - 13x + 36) - 4x + 1$  and so

$$\begin{aligned}
 \frac{2x^4 - 26x^3 + 72x^2 - 4x + 1}{x^2 - 13x + 36} &= \frac{2x^2(x^2 - 13x + 36) - 4x + 1}{x^2 - 13x + 36} \\
 &= 2x^2 + \frac{1 - 4x}{x^2 - 13x + 36} \\
 &= 2x^2 + \frac{1 - 4x}{(x-9)(x-4)}
 \end{aligned}$$

Using partial fractions, we have

$$\frac{1-4x}{(x-9)(x-4)} = \frac{A}{x-9} + \frac{B}{x-4}$$

$$1-4x = A(x-4) + B(x-9)$$

$$\begin{array}{lll} x=4: & -15 = B(-5) & B=3 \\ x=9: & -35 = A(5) & A=-7 \end{array}$$

Hence the integral is

$$\int \left( 2x^2 - \frac{7}{x-9} + \frac{3}{x-4} \right) dx = \frac{2}{3}x^3 - 7 \ln|x-9| + 3 \ln|x-4| + c$$

(Note: you lose half a mark if you forget the absolute value signs and another half if you forget the  $+c$ ).

(b)

Once again, we need to do long division:

$$\begin{array}{r} x \\ x^2 - 16x + 64 \overline{) x^3 - 16x^2 + 68x - 27} \\ \underline{x^3 - 16x^2 + 64x} \phantom{- 27} \\ 4x - 27 \end{array}$$

Hence  $x^3 - 16x^2 + 68x - 27 = x(x^2 - 16x + 64) + 4x - 27$ , so we have

$$\frac{x^3 - 16x^2 + 68x - 27}{x^2 - 16x + 64} = x + \frac{4x - 27}{x^2 - 16x + 64}$$

Note that the denominator factors as  $x^2 - 16x + 64 = (x-8)^2$ . Hence, using partial fractions, we have

$$\frac{4x-27}{x^2-16x+64} = \frac{A}{x-8} + \frac{B}{(x-8)^2}$$

$$4x-27 = A(x-8) + B$$

$$\begin{array}{lll} x=8: & 5 = B \\ x=0: & -27 = A(-8) + 5 & \\ & 27 = 8A - 5 & A=4 \end{array}$$

Thus

$$\begin{aligned} \int \frac{x^3 - 16x^2 + 68x - 27}{x^2 - 16x + 64} dx &= \int x dx + \int \frac{4}{x-8} + \int \frac{5}{(x-8)^2} dx \\ &= \frac{1}{2}x^2 + 4 \ln|x-8| - \frac{5}{x-8} + C \end{aligned}$$

[Note: don't forget the absolute value signs in a logarithm and don't forget the  $+C$ .]

(c)

[3]

Since the degree of the numerator is strictly less than the degree of the denominator, we don't need long division in this case. Note that the denominator is already factored as much as it can be.

Using partial fractions, we have

$$\frac{2x^2 - x - 6}{(x^2 + 4)(x - 1)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 1}$$

$$2x^2 - x - 6 = (Ax + B)(x - 1) + C(x^2 + 4)$$

$$\begin{array}{lll} x = 1 : & -5 = C(5) & C = -1 \\ x = 0 : & -6 = B(-1) - 4 & \\ & 6 = B + 4 & B = 2 \\ x = 2 : & 5 - 2 - 6 = (2A + B)(1) + C(8) & \\ & 0 = 2A + 2 - 8 & \\ & 2A = 6 & \\ & A = 3 & \end{array}$$

The integral is thus

$$\begin{aligned} \int \left( \frac{3x + 2}{x^2 + 4} - \frac{1}{x - 1} \right) dx &= \int \left( \frac{3x}{x^2 + 4} + \frac{2}{x^2 + 4} - \frac{1}{x - 1} \right) dx \\ &= \int \frac{3x}{x^2 + 4} dx + \frac{1}{2} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx - \int \frac{1}{x - 1} dx \end{aligned}$$

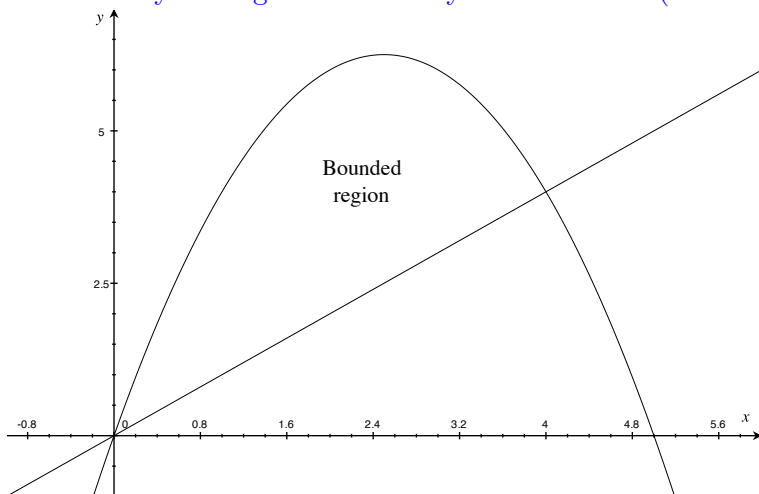
where we divided through by 4 in the second integral. For the first integral, we'll use the substitution  $u = x^2 + 4$  so that  $\frac{du}{dx} = 2x$  or  $dx = \frac{du}{2x}$ . For the second integral, we'll use the substitution  $w = \frac{x}{2}$  so that  $\frac{dw}{dx} = \frac{1}{2}$  or  $dx = 2dw$ . We thus have

$$\begin{aligned} \int \frac{3x}{u} \frac{du}{2x} + \frac{1}{2} \int \frac{1}{w^2 + 1} 2dw - \int \frac{1}{x - 1} dx &= \frac{3}{2} \int \frac{1}{u} du + \int \frac{1}{w^2 + 1} dw - \int \frac{1}{x - 1} dx \\ &= \frac{3}{2} \ln |u| + \arctan w - \ln |x - 1| + c \\ &= \frac{3}{2} \ln |x^2 + 4| + \arctan \left( \frac{x}{2} \right) - \ln |x - 1| + c. \end{aligned}$$

(Note: you lose half a mark if you forget the  $+c$ .)

QUESTION 3. Sketch the region bounded by  $y = x$  and  $y = -x^2 + 5x$ . Find the volume of the object created when this region is rotated around the  $x$ -axis. Leave your answer in exact form (not a decimal approximation).

There is only one region bounded by the two curves (all other possibilities are unbounded):



We need to find the intercepts to determine the limits of integration. This leads to:

$$\begin{aligned}x &= -x^2 + 5x \\x^2 - 4x &= 0 \\x(x - 4) &= 0\end{aligned}$$

and hence the intercepts are 0 and 4. The quadratic is larger than the line, so we have

$$\begin{aligned}V &= \int_0^4 \pi [f_2(x)^2 - f_1(x)^2] dx \\&= \int_0^4 \pi [(-x^2 + 5x)^2 - x^2] dx \\&= \pi \int_0^4 [x^4 - 10x^3 + 24x^2] dx \\&= \pi \left[ \frac{x^5}{5} - \frac{10x^4}{4} + \frac{24x^3}{3} \right]_0^4 \\&= \pi \left( \frac{4^5}{5} - \frac{5}{2} \cdot 4^4 + 8 \cdot 4^3 \right) \\&= \frac{384\pi}{5} \text{ units}^2\end{aligned}$$

QUESTION 4. A patch of moss initially occupies an area of 20 square centimetres. The moss grows at a rate inversely proportional to the size of the patch. After 250 hours, the patch has grown to 30 square centimetres.

- a) Find the size of the patch as a function of time (in hours).
- b) How large is the patch after 300 hours?
- c) How long does it take until the moss reaches 50 square centimetres?

[3] (a) The differential equation that describes this phenomenon is

$$\frac{dP}{dt} = \frac{k}{P}$$

Separating variables, we have

$$\begin{aligned} PdP &= kdt \\ \int PdP &= \int kdt \\ \frac{1}{2}P^2 &= kt + c \\ P &= \sqrt{2kt + 2c} \end{aligned}$$

(choosing the positive root since moss can't have a negative area). Applying the initial condition, we have

$$\begin{aligned} 20 &= \sqrt{2k(0) + 2c} \\ 400 &= 2c \\ c &= 200 \end{aligned}$$

After 250 hours, we have

$$\begin{aligned} 30 &= \sqrt{2k(250) + 400} \\ 900 &= 500k + 400 \\ k &= 1 \\ P(t) &= \sqrt{2t + 400} \end{aligned}$$

[1] (b) After 300 hours, we have

$$P(300) = \sqrt{2(300) + 400} = \sqrt{1000} = 31.6 \text{ cm}^2$$

[Note: you lose half a mark if you don't specify the units.]

(c)

[1]

We have

$$\begin{aligned}50 &= \sqrt{2\bar{t} + 400} \\2500 &= 2\bar{t} + 400 \\ \bar{t} &= 1050 \text{ hours}\end{aligned}$$

[Note: you lose half a mark if you don't specify the units.]