TOPIC 5. INTRODUCING VECTORS

1. GEOMETRIC VECTORS

Read §§ 6.1-6.5

A. Geometric Vector, Its Magnitude and Direction

A (2- or 3- dimensional) *vector* is a quantity that has both *magnitude* and *direction*.

A *geometric vector* is represented by an arrow from a point *A* to another point *B* denoted by $\overrightarrow{AB}$, where *A* the *tail* of the vector, and *B* is the *head* of the vector. Without specifying the head and the tail, using a single symbol to denote a vector we use an arrow on top the symbol such as $\vec{v}$ or in boldface such as **v**.

The magnitude of a vector $\mathbf{v} = \overrightarrow{AB}$ is the length of the line segment *AB*, denoted by $|\mathbf{v}|$ or $|\overrightarrow{AB}|$. (Note that, in some book, the magnitude of a vector **v** is denoted by $\|\mathbf{v}\|$, and call this number the *norm* of **v**). If *A* and *B* are two identical points, then the magnitude of this vector is 0. This is the *zero vector*, denoted by **0** or $\vec{0}$. The direction of the zero vector is undefined. If the magnitude of a vector is 1, then this vector is a *unit vector*.

The direction of a 2-dimensional vector $\overrightarrow{AB}$ is specified by an angle. This angle may be measured in different ways:

In mathematics, the angle is measured in a Cartesian coordinate system with the tail of the vector as the origin. The angle is measured **in Radians** from the positive direction of the *x*-axis to the line *AB* counter-clockwise.

In navigation, vector directions are expressed as *bearings* measured **in degrees**.

(a) A (*true*) *bearing* (or *azimuth bearing*) is a compass measurement where the angle is measured from north clockwise.

True bearings are expressed as three-digit numbers, including leading zeros. For example, a bearing of 040° is an angle of 40° clockwise from due north.

(b) *Quadrant bearing* is a measurement between 0° and 90° east or west of the north-south line. The quadrant bearing N23°W is read as 23° west of north, whereas S20°E is read as 20° east of south. All quadrant bearings are referenced from north or south, not from west or east.

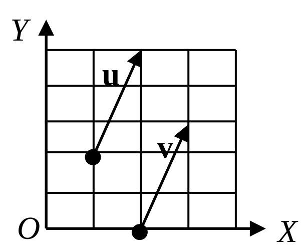
Example 5.1.1. a. A direction given by $\frac{\pi}{3}$ radians in mathematics is 030° in true bearing, or N30°E in quadrant bearing.

b. A direction specified by 315° in true bearing is $\frac{5\pi}{4}$ in mathematics, or N45°W in quadrant bearing.

c. A direction specified S60°E in quadrant bearing is $-\frac{\pi}{6}$ (or $\frac{11\pi}{6}$) in mathematics, or 120° in true bearing.

Two vectors are *parallel* (also called *collinear*) if they have the same or opposite directions. Two vectors **u** and **v** are *equivalent* (or *equal*), denoted by **u** = **v**, if they have the same direction and the same magnitude. By this definition, we can identify all vectors with the same direction and the same magnitude. If we identify all vectors with the same direction and same magnitude (regardless the position of their tails), then we have *position vectors*.

Example 5.1.2. Let **u** be a vector with tail (1, 2) and head (2, 5). Let **v** be a vector with tail (2, 0) and head (3, 3). They are represented by different arrows as in the following figure. However, as position vectors, since **u** and **v** have the same direction and same magnitude, they are regarded as the same vector, i.e., **u** = **v**.



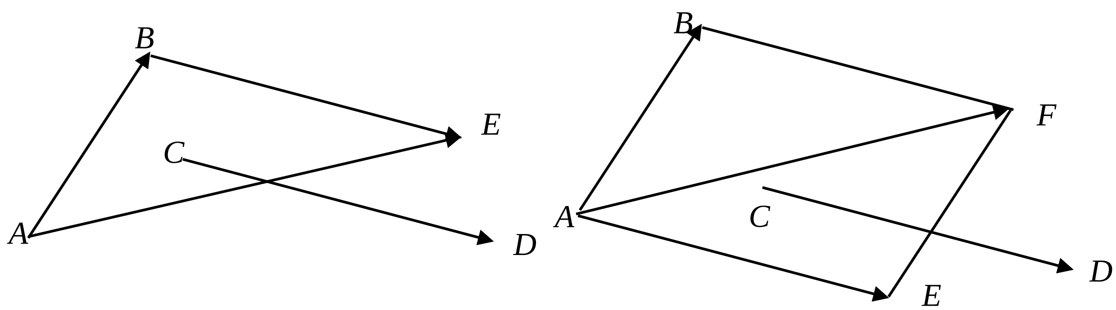
From now on, we will say "a vector" to mean "a position vector" unless we want to emphasize.

B. Operations of Geometric Vectors

Operation *Addition* is defined to find the *sum* of two vectors **u** and **v**, denoted by **u** + **v**. To find the sum of two vectors, we can use the *triangle method* or the *parallelogram method*.

The Triangle Method: Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two vectors. Draw a vector $\overrightarrow{BE}$ that is equivalent to $\overrightarrow{CD}$ with *B* as the tail (so that $\overrightarrow{BE} = \overrightarrow{CD}$). Then $\overrightarrow{AE}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{CD}$.

The Parallelogram Method: Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two vectors. Draw a vector $\overrightarrow{AE}$ that is equivalent to $\overrightarrow{CD}$ with *A* as the tail (so that $\overrightarrow{AE} = \overrightarrow{CD}$). Construct a parallelogram *AEFB* with *AB* and *AE* as two adjacent sides. Then $\overrightarrow{AF}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, $\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{CD}$.



Triangle method   Parallelogram method

In particular, If **u** and **v** are of the same direction, then **u** + **v** have the same direction with magnitude | **u** | + | **v** |. If **u** and **v** are of opposite direction, then | **u** + **v** | = | | **u** | − | **v** | |, i.e., the absolute value of | **u** | − | **v** |, and the direction of **u** + **v** is the same as the one between **u** and **v** that has the larger magnitude.

Operation *scalar multiplication* defines the *product* of a vector **v** and a scalar *c* (i.e., a number). Denoted by *c***v**.

If *c* is positive, *c***v** has the same direction as **v**, and | *c***v** | = *c* | **v** |. If *c* is negative. *c***v** has the opposite direction of **v**, and | *c***v** | = | *c* | | **v** |.

Two non-zero vectors **u** and **v** are collinear if and only if **u** = *c***v** for some $c \neq 0$.

The product (−1)**v** is called the *negation* of **v**, denoted by –**v**. Use the negation of a vector, we can define the operation *subtraction* that finds the *difference* of two vectors **u** and **v**, denoted by **u** – **v**. The difference **u** – **v** = **u** + (−**v**).

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a set of vectors. A *linear combination* of these vectors is an expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, where $c_1, c_2, \ldots, c_n$ are scalars, called *coefficients*.

Properties of Addition and Scalar Multiplication of Vectors

Let **u**, **v** and **w** be three vectors, and let c and d be two numbers. Then

u + **v** = **v** + **u** (the commutative law of addition)
(**u** + **v**) + **w** = **u** + (**v** + **w**) (the associative law of addition)
u + **0** = **u**
u + (−**u**) = **0**
$c(d\mathbf{u}) = (cd)\mathbf{u}$ (the associative law of scalar multiplication)
$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (the distributive law of scalar multiplication with respect to addition)
$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (the distributive law of addition with respect to scalar multiplication)
1**u** = **u**
(−1)u = −**u**
0u = **0**

Using these properties, we can simplify expressions of vectors.

Example 5.1.3. Let **u**, **v**, and **w** be vectors. Then

$2(\mathbf{u} + 2\mathbf{v} - w) - (4\mathbf{u} + 3\mathbf{v} + 2\mathbf{w}) = 2\mathbf{u} + 4\mathbf{v} - 2\mathbf{w} - 4\mathbf{u} - 3\mathbf{v} - 2\mathbf{w} = -2\mathbf{u} + \mathbf{v} - 4\mathbf{w}$.

C. Applications

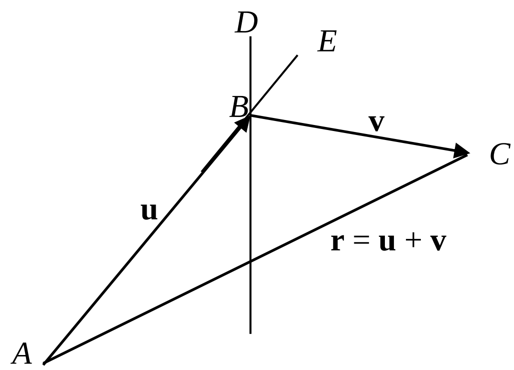
Resultant and Equilibrant

Inspired by application of vectors in mechanics, the sum of a set of two or more vectors is called the *resultant* of this set. The negation of the resultant is called the *equilibrant*.

The resultant and the equilibrant of two vectors can be found by the Laws in trigonometry. In particular, the resultant and equilibrant of two perpendicular vectors can be found by Pythagorean Theorem.

Example 5.1.4. Let **u** and **v** be two 2-dimensional vectors. | **u** | = 3 and | **v** | = 2. The direction of **u** is 040° and the direction of **v** is 100° in true bearing. Find the resultant and equilibrant of **u** and **v**.

Solution. See the following figure:



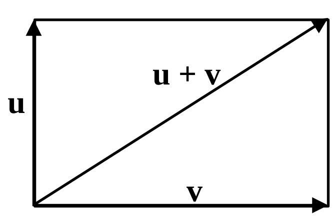
By the conditions given in the problem, $\angle DBE = 40°$ and $\angle DBC = 100°$. Hence, $\angle EBC = 60°$, and $\angle ABC = 120°$. The resultant is $\mathbf{r} = \mathbf{u} + \mathbf{v} = \overrightarrow{AC}$. By the Cosine Law, $|\,\mathbf{r}\,|^2 = |\,\mathbf{u}\,|^2 + |\,\mathbf{v}\,|^2 - 2|\,\mathbf{u}\,||\,\mathbf{v}\,|\cos 120°$. Since $\cos 120° = \cos(180° - 120°) = -\cos 60° = -\frac{1}{2}$, $|\,\mathbf{r}\,|^2 = 9 + 4 + 6 = 19$. $|\,\mathbf{r}\,| = \sqrt{19}$.

By the Sine Law, $\frac{\sin ABC}{|\,AC\,|} = \frac{\sin BAC}{|\,\mathbf{v}\,|}$. Hence, $\sin BAC = \frac{\sqrt{3}}{\sqrt{19}} \approx 0397$. $\angle BAC \approx 23.4°$.

The direction of the resultant is 40 + 23.4 = 63.4° in true bearing, and the magnitude of the resultant is $\sqrt{19}$.

The equilibrant has the same magnitude $\sqrt{19}$, and the direction is 63.4 + 180 = 243.4° in true bearing.

Example 5.1.5. One force **u** acting on an object straight up is of magnitude 3 Newtons, and another force **v** acting on the same object horizontally in the positive direction of the *x*-axis is of magnitude 4 Newtons. Find the resultant and the equilibrant.



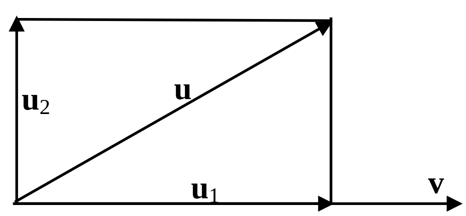
Solution. By the Pythagorean Theorem, the magnitude of the resultant is $\sqrt{3^2+4^2}$ = 5 Newtons. The direction of the resultant is specified by the angle θ with tan $\theta = \frac{3}{4}$. Then $\theta \approx 0.64$ (radian).

The equilibrant has the same magnitude as the resultant, i.e., 5 Newtons, and the direction is specified by the angle 0.64 + π = 3.79 (radians).

Resolution of Vectors into Rectangular Components

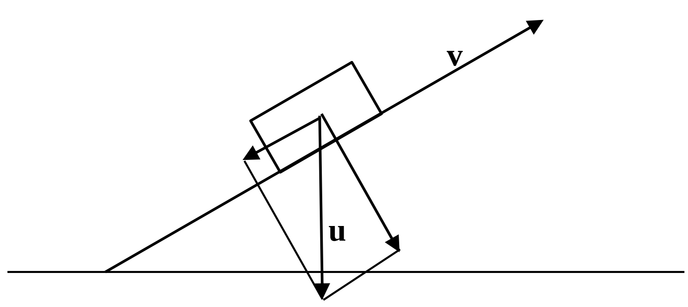
In many applications of physics, want to decompose a force into two perpendicular components. These questions can be formulated in the following way:

Let **u** and **v** be two vectors. We want to find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is collinear to **v** and $\mathbf{u}_2$ is perpendicular to **v**. Vector $\mathbf{u}_1$ is called the *projection* of **u** onto **v**, denoted by $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.



Let θ be the angle between **u** and **v**. Then the magnitude of $\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u}$ is $|\,\text{proj}_{\mathbf{v}}\,\mathbf{u}\,| = |\,\mathbf{u}\,| \cos\theta$. The direction of $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is the same as the direction of **v** if $\theta < \pi/2$, and the direction of $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is opposite to the direction of **v** if $\theta > \pi/2$. Obviously, if **u** is perpendicular to **v**, the $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is the zero vector **0**. On the other hand, the magnitude of $\mathbf{u}_2$ is $|\,\mathbf{u}_2\,| = |\,\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}\,| = |\,\mathbf{u}\,| \sin\theta$.

Example 5.1.6. Suppose a box of weight 120 kg is put on an inclined surface at an angle of π / 6. Find the force, in Newtons, against the surface and the force that pulls the box down along the surface.



Let **v** be a vector in the direction of the inclined surface with angle $\pi / 6$. Let **u** be the vector that represents the weight of the box as a force with vertically downward direction. Then the angle between **u** and **v** is $\pi / 6 + \pi / 2 = 2\pi / 3$. Take the acceleration of gravity $g \approx 9.81$ m / sec^2. The force collinear to **v** that pulls the box down along the surface is $120g \cos (2\pi / 3) \approx -588.6$ Newtons, (the negative sign means that the direction is opposite to the direction of **v**), and the force against the surface is $120g \sin (2\pi / 3) \approx 1019.5$ Newtons.

2. CARTESIAN VECTORS

Read §§ 7.1-7.6

A. 2- or 3-Dimensional Cartesian Vectors

A position vector is determined uniquely by its magnitude and direction. Hence, we may assume that the tails of position vectors are at the origin. Let **u** be a position vector on the Cartesian plane, i.e., a plane with a Cartesian coordinate system, with the origin as its tail. Its head is at a point (a, b). Then we can establish a *one-to-one correspondence* relation between the position vectors and the points on the plane, i.e., every vector corresponds to a unique point, and every point corresponds to a unique position vector. Therefore, we can identify position vectors with points on a Cartesian plane.

On the other hand, since every point on a Cartesian plane has a unique ordered pair of real numbers (x, y) as its coordinates, and every ordered pair of real numbers (x, y) corresponds to a unique point on a Cartesian plane, we may also identify the points on a Cartesian plane with all ordered pair of real numbers.

Similarly, a position vector in a Cartesian space, i.e., a plane with a 3-dimensional Cartesian coordinate system, can be identified with a point in the pace, or an ordered triple of real numbers.

In this way, we may use an ordered pair or an ordered triple of real numbers to represent a position vector. A position vector represented by an ordered pair or an ordered triple of real numbers is called a *Cartesian vector*.

Because the set of position vectors, the set of points on a Cartesian plane (or in a Cartesian space), and the set of ordered pairs (or ordered triples) of real numbers, are identified, we will say "a vector" to mean either a geometric position vector, or a point on a Cartesian plane (or in a Cartesian space), or an ordered pair (or an ordered triple) of real numbers.

The set of all 2-dimensional vectors is denoted by $\mathbb{R}^2$, and the set of all 3-dimensional vectors is denoted by $\mathbb{R}^3$.

In $\mathbb{R}^2$, the vector (1, 0) is called the *unit vector* in the direction of the x-axis, denoted by **i** or $\vec{i}$, the vector (0, 1) is called the *unit vector* in the direction of the y-axis, denoted by **j** or $\vec{j}$.

When a vector (a, b) is decomposed into a vector in the direction of the x-axis, and a vector in the direction of the y-axis, we have

$(a, b) = a\,\mathbf{i} + b\,\mathbf{j}$.

The magnitude of vector $\mathbf{u} = (a, b)$ is $|\,\mathbf{u}\,| = \sqrt{a^2 + b^2}$.

In $\mathbb{R}^3$, the vector (1, 0, 0) is called the *unit vector* in the direction of the x-axis, denoted by **i** or $\vec{i}$, the vector (0, 1, 0) is called the *unit vector* in the direction of the y-axis, denoted by **j** or $\vec{j}$, the vector (0, 0, 1) is called the *unit vector* in the direction of the z-axis, denoted by **k** or $\vec{k}$.

When a vector (a, b, c) is decomposed into a vector in the direction of the x-axis, a vector in the direction of the y-axis, and a vector in the direction of the z-axis, we have

$(a, b, c) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$.

The magnitude of vector $\mathbf{u} = (a, b, c)$ is $|\,\mathbf{u}\,| = \sqrt{a^2 + b^2 + c^2}$.

B. Addition and Multiplication of Vectors

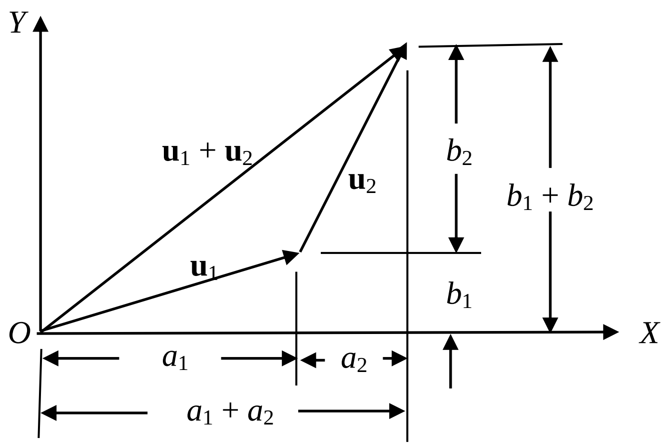
In $\mathbb{R}^2$, the sum of two vectors $\mathbf{u}_1 = (a_1, b_1)$ and $\mathbf{u}_2 = (a_2, b_2)$ is

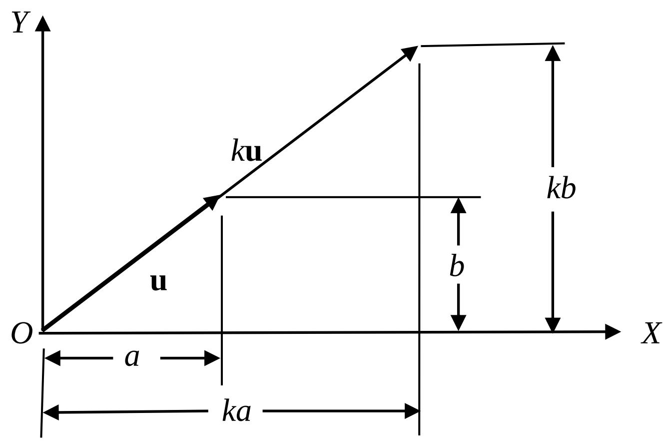
$\mathbf{u}_1 + \mathbf{u}_2 = (a_1 + a_2, b_1 + b_2)$.

The product of $\mathbf{u} = (a, b)$ and a scalar k is

$k\mathbf{u} = (ka, kb)$.

These results can be justified by the following diagrams:





Similarly, in $\mathbb{R}^3$, the sum of two vectors $\mathbf{u}_1 = (a_1, b_1, c_1)$ and $\mathbf{u}_2 = (a_2, b_2, c_2)$ is

$\mathbf{u}_1 + \mathbf{u}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$.

The product of $\mathbf{u} = (a, b, c)$ and a scalar k is

$k\mathbf{u} = (ka, kb, kc)$.

Therefore, the difference of two vectors are given by

$(a_1, b_1) - (a_2, b_2) = (a_1, b_1) + (-1)(a_2, b_2) = (a_1, b_1) + (-a_2, -b_2) = (a_1 - a_2, b_1 - b_2)$, and

$(a_1, b_1, c_1) - (a_2, b_2, c_2) = (a_1, b_1, c_1) + (-1)(a_2, b_2, c_2) = (a_1, b_1, c_1) + (-a_2, -b_2, -c_2)$
$= (a_1 - a_2, b_1 - b_2, c_1 - c_2)$.

A vector **u** with tail **a** = (x_1, y_1) and head **b** = (x_2, y_2), as a position vector, equals the vector $(x_2 - x_1, y_2 - y_1)$ = **b** – **a**. Note that **a** and **b** are regarded as both position vectors and points. The magnitude of **u** is | **u** | = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Similarly, in $\mathbb{R}^3$, a vector **u** with tail **a** = (x_1, y_1, z_1) and head **b** = (x_2, y_2, z_2) equals the vector **b** –**a** = $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. The magnitude of **u** is | **u** | = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

C. The Dot Product

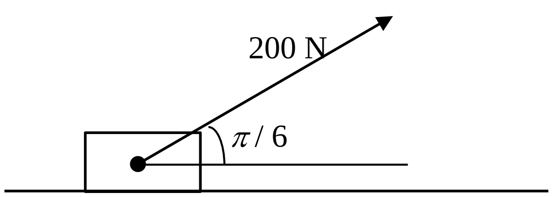
A Motivation – Work Done by a Force

If a force $\boldsymbol{F}$ acting on an object causes it to move in a direction **v** for a distance s, then the work done by this force is the product of the projection of $\boldsymbol{F}$ onto the direction of the movement multiplied by the distance s:

$W = s \operatorname{proj}_{\mathbf{v}} \boldsymbol{F}$.

Let **s** be the vector in the direction of the movement with magnitude s, and let θ be the angle between the force and vector **s**. Then the work is $W = |\mathbf{s}|\,|\boldsymbol{F}| \cos\theta$.

Example 5.2.1. A man pulls a box on the ground with a force 200 Newtons in an angle $\pi/6$. If the box moves for 10 meters, find the work done, in Joules.



Solution. The magnitude of the projection of $\boldsymbol{F}$ onto **s** is $\boldsymbol{F}_1 = 200\cos(\pi/6) = 100\sqrt{3}$. The work done is $\boldsymbol{F}_1 \times |\mathbf{s}| = 1000\sqrt{3}$ Joules.

Definition

The *dot product* of two vectors **u** and **v**, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined to be

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between **u** and **v, $0 \le \theta \le \pi$**. When $\theta = 0$, **u** and **v** are in the same direction, and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$. When $\theta = \pi$, **u** and **v** are in the opposite directions, and $\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$. When $\theta = \pi/2$, **u** and **v** are perpendicular, and $\mathbf{u} \cdot \mathbf{v} = 0$.

Note that, because $\cos \boldsymbol{\theta} = \cos(-\boldsymbol{\theta})$, it doesn't matter if we measure this angle from **u** to **v** or from **v** to **u**.

Using the dot product, the work done by a force $\boldsymbol{F}$ acting on an object that causes the object to move for a vector **s** is the dot product $W = \boldsymbol{F} \cdot \mathbf{s}$.

We can find the angle between two vectors using the dot product. Indeed,

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Note that $|\mathbf{u}| \cos \theta$ is the magnitude of the projection of **u** onto **v**. Then we also have

$$\mathbf{u} \cdot \mathbf{v} = |\operatorname{proj}_{\mathbf{v}} \mathbf{u}||\mathbf{v}|, \text{ or}$$

$$|\operatorname{proj}_{\mathbf{v}} \mathbf{u}| = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Since the projection of **u** onto **v** is collinear with vector **v**,

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \times \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}.$$

The dot product is also called the *scalar product* because the result of the dot product of two vectors is a scalar.

Calculating the Dot Product

Let $\mathbf{u}_1 = (a_1, b_1)$ and $\mathbf{u}_2 = (a_2, b_2)$. Then $\mathbf{u}_2 - \mathbf{u}_1 = (a_2 - a_1, b_2 - b_1)$. Using the triangle method, we see that $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_2 - \mathbf{u}_1$ form a triangle. By the cosine law,

$$|\mathbf{u}_2 - \mathbf{u}_1|^2 = |\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 - 2|\mathbf{u}_1||\mathbf{u}_2| \cos \theta.$$

Hence,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = \frac{1}{2}(|\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 - |\mathbf{u}_2 - \mathbf{u}_1|^2) = \frac{1}{2}((a_1{}^2 + b_1{}^2) + (a_2{}^2 + b_2{}^2) - (a_2 - a_1)^2 + (b_2 - b_1)^2)$$

$$= \frac{1}{2}(2a_1a_2 + 2b_1b_2) = a_1a_2 + b_1b_2.$$

Similarly, in $\mathbb{R}^3$, we can show $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = a_1a_2 + b_1b_2 + c_1c_2$.

Properties of the Dot Product

Let **u, v** and **w** be vectors, and let *c* and *d* be scalars.

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. (commutative law)

b. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$.

c. $\mathbf{u} \cdot (\mathbf{v} + \mathrm{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

d. $\mathbf{u} \cdot \mathbf{0} = 0$.

e. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

Example 5.2.2. Show that the vectors $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (1, 2, 1)$ are perpendicular.

Solution. The dot product $\mathbf{u} \cdot \mathbf{v} = 3 \times 1 + (-2) \times 2 + 1 \times 1 = 0$. Hence, **u** and **v** are perpendicular.

Example 5.2.3. Find the angle between vectors $\mathbf{u} = (2, -1, 2)$ and $\mathbf{v} = (-2, 3, 6)$.

Solution. $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{-4-3+12}{3\times7} = \dfrac{5}{21}$. $\theta \approx 1.33$ (Radians).

Example 5.2.4. Let $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (3, -6, 0)$. Find the projection of **u** onto **v**.

$$\mathrm{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{(-3,4,1)\cdot(1,-2,-1)}{6}(1,-2,-1) = \frac{-3-8-1}{6}(1,-2,-1) = (-2, 4, 2).$$

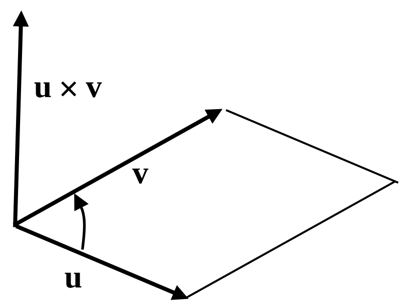
D. The Cross Product of 3-Dimensional Vectors

Definition

Note that the cross product is only defined in $\mathbb{R}^3$.

Let $\mathbf{u} = (x_1, x_2, x_3)$, $\mathbf{v} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$.

The *cross product* of **u** and **v**, denoted by $\mathbf{u} \times \mathbf{v}$ is defined to be a vector in $\mathbb{R}^3$ whose magnitude equals the area of the parallelogram with **u** and **v** as two adjacent sides, and whose direction is perpendicular to the plane spanned by **u** and **v** with the positive direction determined by the right-hand rule.



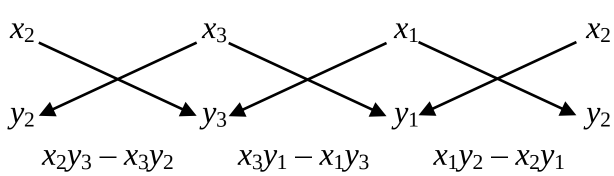
By this definition, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the angle between **u** and **v**. If **u** and **v** are collinear, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$; if **u** and **v** are perpendicular, then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

For unit vectors along the axes, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

We can prove that, with the Cartesian form, the cross product

$\mathbf{u} \times \mathbf{v} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

To memorize the definition of the cross product, we may use the following diagram:



Example 5.2.5. $(1, 2, 3) \times (4, 5, 6) = (-3, 6, -3)$.

Properties of the Cross Product

Let $\mathbf{u} = (x_1, x_2, x_3)$ and $\mathbf{v} = (y_1, y_2, y_3)$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

a. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

b. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

c. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
$(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$, (The distributive law)

d. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. (The cross product does not satisfy the commutative law!)

e. $(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v})$.

Note that the cross product is neither commutative nor associative.

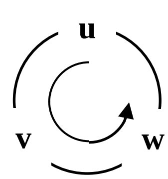
E. Mixed Product

Definition

The mixed product (also called the *triple scalar product* as in the textbook) of three vector **u**, **v** and **w** is

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$.

Since the cross product does not satisfy the commutative law, the order of the factors in a mixed product is significant. The following diagram helps to memorize these three equivalent forms:

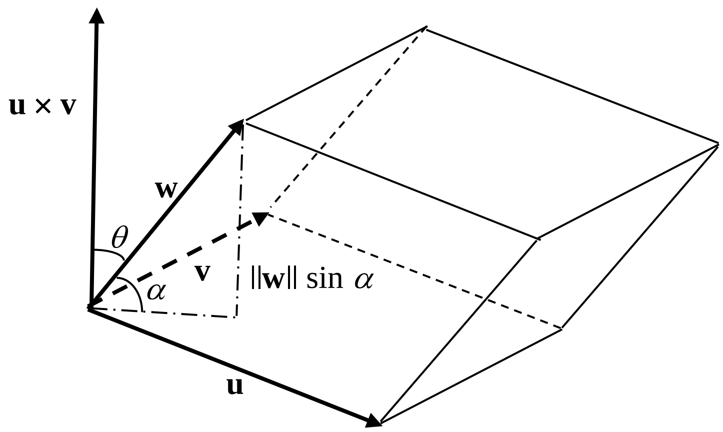


If rotate the factors in the other direction, the mixed product changes its sign:

$(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = (\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} = -(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

The absolute value of the mixed product $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of the parallelepiped spanned by u, v, and w.

Indeed, $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathrm{v}||\mathbf{w}| \cos\theta$, where θ is the angle between $\mathbf{u} \times \mathrm{v}$ and **w**. The area of the base of the parallelepiped spanned by **u**, **v**, and **w** is $A = |\mathbf{u} \times \mathbf{v}|$, and the height of this parallelepiped is $h = |\mathbf{w}| \sin\alpha$, where α is the angle between **w** and the base of the parallelepiped. Since $\alpha = \pi/2 - \theta$, $\cos\theta = \sin\alpha$. The volume of this parallelepiped in the parallelepiped $V = Ah = |\mathbf{u} \times \mathrm{v}||\mathbf{w}| \sin\alpha = |\mathbf{u} \times \mathrm{v}||\mathbf{w}| \cos\theta$,



By this definition, vectors **u**, **v**, and **w,** are *coplanar* (i.e., in the same plane) if and only if the mixed product is 0.

Example 5.2.6. Find the volume of the parallelepiped spanned by vectors **u** = (1, 0, 1), **v** = (1, 2, 2), and **w** = (1, −1, 1).

Solution. **u** × **v** = (−2, −1, 2), (**u** × **v**) · **w** = −2 + 1 + 2 = 1.

Example 5.2.7. Show that vectors **u** = (1, 0, 1), **v** = (1, 2, 2), and **w** = (−2, 2, −1) are coplanar.

Solution. **u** × **v** = (−2, −1, 2), (**u** × **v**) · **w** = 4 −2 −2 = 0. These three vectors are coplanar.

TOPIC 6. LINES AND PLANES

Read §§ 6.1-6.6

1. LINES

A. Equations of Lines

In $\mathbb{R}^2$ or $\mathbb{R}^3$, if a line is parallel to a vector **v** and going through a point $\mathbf{p}_0$, then every point **p** on this line can be expressed as $\mathbf{p}_0 + t\,\mathbf{v}$, where t is a real number (*parameter*). This line can be expressed as $\mathbf{p} = \mathbf{p}_0 + t\mathbf{v}$. This is called the *vector form* of the equation of the line and **v** is the *directional vector* of this line, which specifies the direction of this line.

In $\mathbb{R}^2$, write $\mathbf{p} = (x, y)$, $\mathbf{p}_0 = (x_0, y_0)$ and $\mathbf{v} = (a, b)$. Then

$(x, y) = (x_0, y_0) + t(a, b)$, or

$x = x_0 + at,$
$y = y_0 + bt.$

This is the *parametric form* of this line.

If we eliminate the parameter t from these two equations, we can express the equation of a line in the ordinary *scalar form* as a linear equation.

Similarly, in $\mathbb{R}^3$, the parametric form of a line is

$x = x_0 + at,$
$y = y_0 + bt,$
$z = z_0 + ct,$

where $\mathbf{p} = (x, y, z)$ is a general point on this line, $\mathbf{p}_0 = (x_0, y_0, z_0)$ is an arbitrary particular point on this line, and $\mathbf{v} = (a, b, c)$ is the directional vector of this line.

Note that, since we may choose different points on this line, and use a non-zero multiple to replace the current directional vector, the vector form and the parametric form are not unique.

Example 6.1.1. A line in $\mathbb{R}^2$ has the equation $2x - y = 1$. Find the vector form of this line.

Solution. Since the points $\mathbf{p}_1 = (1, 1)$ and $\mathbf{p}_2 = (2, 3)$ are on this line, a directional vector is $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1 = (1, 2)$. The vector form of this line is $\mathbf{p} = (1, 1) + t\,(1, 2)$.

The parametric form of this line is $x = 1 + t$ and $y = 1 + 2t$. If we eliminate the parameter t from this system, we get back to $2x - y = 1$.

Example 6.1.2. Find the parametric form of a line going through two points $\mathbf{p}_1 = (2, 1, -3)$ and $\mathbf{p}_2 = (0, 2, -1)$.

Solution. This line is parallel to the vector $\mathbf{v} = \mathbf{p}_1 - \mathbf{p}_2 = (2, -1, -2)$. The vector form of this line is $\mathbf{p} = \mathbf{p}_1 + t\,\mathbf{v}$. Then $(x, y, z) = (2, 1, -3) + t\,(2, -1, -2)$, and the parametric form of this line is

$$\begin{cases} x = 2 + 2t, \\ y = 1 - t, \\ z = -3 - 2t. \end{cases}$$

B. Relations between Two Lines

In $\mathbb{R}^2$, two lines may be identical, parallel, or intersect.

In $\mathbb{R}^3$, two lines may be identical, parallel, intersect, or skew. Two lines are *skew* if there are not in the same plane.

If the directional vectors of two lines are collinear, then these two lines are either identical or parallel. In this case, if they have a common point, they are identical; otherwise, they are parallel. In $\mathbb{R}^2$, if two lines are not identical or parallel, they have an intersection. In $\mathbb{R}^3$, if two lines are not identical or parallel, they may have an intersection, or they are skew lines.

Example 6.1.3. Find the intersection of lines $x + 2y = 1$, and $2x + 3y = 4$.

Solve the system of equations:

$x + 2y = 1$,
$2x + 3y = 4$.

From the first equation, $x = 1 - 2y$. Plug this into the second equation: $2(1 - 2y) + 3y = 4$, or $2 - y = 4$. $y = -2$. Then $x = 1 - 2 \times (-2) = 5$. These two lines intersect at the point $(5, -2)$.

Example 6.1.4. Show that the lines $\mathbf{p} = (1, -2, 4) + t\,(1, -3, -1)$ and $\mathbf{p} = (0, 3, -3) + s\,(2, 1, 4)$ are skew lines.

Since the directional vectors of these two lines are not collinear, these two lines cannot be identical or parallel. Try to find the intersection of these two lines. If we can find an intersection, then these two lines intersect. If we cannot find an intersection point, then they are skew lines. (Note that we use different symbols to represent the parameters in these lines).

The parametric forms of these two lines are

$x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$.

$x = 2s$, $y = 3 + s$, $z = -3 + 4s$.

If they have an intersection, we have a value of t and a value of s, such that $1 + t = 2s$, $-2 + 3t = 3 + s$, and $4 - t = -3 + 4s$.

From the first equation, we have $t = 2s - 1$. Plugging this into the second equation, we find $s = 8 / 5$, and plugging in this into the third equation, we find $s = 4 / 3$. This is a contradiction. Therefore, these two lines are skew lines.

2. PLANES

Read §§ 8.2, 8.3, 8.5, 8.6

A. The Equation of a Plane

In $\mathbb{R}^3$, the graph of the equation $ax + by + cz = d$ is a *plane*.

If $a = 0$, this plane is parallel to the x-axis and perpendicular to the y-z plane. If $b = 0$, this plane is parallel to the y-axis and perpendicular to the x-z plane. If $c = 0$, this plane is parallel to the z-axis and perpendicular to the x-y plane.

If $a = b = 0$ and $c \neq 0$, this plane is perpendicular to the z-axis and parallel to the x-y plane. If $a = c = 0$ and $b \neq 0$, this plane is perpendicular to the y-axis and parallel to the x-z plane. If $b = c = 0$ and $a \neq 0$, this plane is perpendicular to the x-axis and parallel to the y-z plane.

If all a, b, and c, are not zero, this plane interests the x-axis at the point $(d / a, 0, 0)$, it intersects the y-axis at $(0, d / b, 0)$, and it intersects the z-axis at $(0, 0, d / c)$.

The x-coordinate of the intersection of a plane and the x-axis is called the *x-intercept* of this plane. The y-coordinate of the intersection of a plane and the y-axis is called the *x-intercept* of this plane. The z-coordinate of the intersection of a plane and the x-axis is called the *z-intercept* of this plane.

If $d = 0$, this plane goes through the origin, and all intercepts are 0.

Let $\mathbf{p}_0 = (x_0, y_0, z_0)$ be a point on a plane P with equation $ax + by + cz = d$, i.e., $ax_0 + by_0 + cz_0 = d$. Let $\mathbf{p} = (x, y, z)$ be an arbitrary point on P. Then $ax + by + cz = ax_0 + by_0 + cz_0$, or

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

Using the dot product, we have $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$, where $\mathbf{n} = (a, b, c)$. Since $\mathbf{v} = \mathbf{p} - \mathbf{p}_0$ is a vector on P, vector $\mathbf{n}$ is perpendicular to $\mathbf{v}$. Choose different point $\mathbf{p}$ on this plane, we see that vector $\mathbf{n}$ is perpendicular to every vector on P. In other words, $\mathbf{n}$ is perpendicular to P. A non-zero vector perpendicular to a plane is called a *normal vector* of the plane. Vector $\mathbf{v}$ is a normal vector of the plane $ax + by + cz = d$ if and only if $\mathbf{v} = k\mathbf{n}$, where k is a non-zero number and $\mathbf{n} = (a, b, c)$.

In particular, if $ax + by + cz = d$ is the equation of a plane P, then (a, b, c) is a normal vector of P. Conversely, if (a, b, c) is a normal vector of a plane P, then the equation of P has the form $ax + by + cz = d$.

If a vector $\mathbf{n} = (a, b, c)$ and a point $\mathbf{p}_0 = (x_0, y_0, z_0)$ is given, the plane with $\mathbf{n}$ as a normal vector and contains point $\mathbf{p}_0$ can be expressed by the vector equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$, or $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$. This is called the *normal-point form* of the plane. Because we may choose different points on this lane, and use different normal vectors, the normal-point form is not unique.

Example 6.2.1. Find the equation of a plane in $\mathbb{R}^3$ with a normal $\mathbf{n} = (2, 3, -6)$ containing a point $\mathbf{p}_0 = (4, -2, 1)$.

Solution. The equation is $(2, 3, -6) \cdot (x - 4, y + 2, z - 1) = 0$, or $(2x - 8) + (3y + 6) + (-6z + 6) = 0$, or $2x + 3y - 6z + 4 = 0$.

Example 6.2.2. Write the plane $2x - y + 3z = 4$ in a normal-point form.

Solution. A normal vector of this plane is $\mathbf{n} = (2, -1, 3)$, whose components are the coefficients of the variables in the scalar form. There are many different ways to find a point on the plane.

For instance, we let $x = 0$ and $z = 0$. Then $y = -4$. We have point $\mathbf{p}_0 = (0, -4, 0)$ on this plane. A normal-point form of this plane is $(2, -1, 3)\cdot(x, y + 4, z) = 0$.

Example 6.2.3. Find the scalar equation of the plane that contains three points $\mathbf{p}_1 = (1, 2, 3)$, $\mathbf{p}_2 = (1, 0, 0)$, and $\mathbf{p}_3 = (0, 1, 1)$.

Solution. Since $\mathbf{u} = \mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_3$, are on the plane, $\mathbf{u} \times \mathbf{v} = (-1, -3, 2)$ is a normal vector of the plane. In other words, the equation of the plane has the form $-x - 3y + 2z = d$. Since $\mathbf{p}_2 = (1, 0, 0)$ is on the plane, $d = -1$. The equation of the plane is $-x - 3y + 2z = -1$.

Example 6.2.4. Find the equation of a plane P that is parallel to the line L_1: $(x, y, z) = (0, 0, 0) + t\,(1, 2, -3)$, and contains the line L_2: $(x, y, z) = (6, 1, 2) + s\,(2, -1, 3)$.

Solution. The normal vector of P is perpendicular to the directional vectors of L_1 and L_2. Find a normal vector of P by the cross product

$\mathbf{n} = (1, 2, -3) \times (2, -1, 3) = (3, -9, -5)$. The equation of the P has the form $3x - 9y - 5z = d$,

Since point (6, 1, 2) is on L_2 and P contains L_2, $d = 3 \times 6 - 9 \times 1 - 5 \times 2 = -1$, and the equation of P is $3x - 9y - 5z = -1$.

B. Intersections

If the directional vector of a line L is perpendicular to a normal vector of a plane P, then L is either in P, or it is parallel to P. Take an arbitrary point $\mathbf{p}$ in L. If $\mathbf{p}$ is in P, then L is in P, otherwise, L is parallel to P.

If the directional vector of a line L is not perpendicular to a normal vector of a plane P, then they have a unique intersection point.

If a normal vector $\mathbf{n}_1$ of a plane P_1 and a normal vector $\mathbf{n}_2$ of a plane P_2 are collinear, then P_1 and P_2 are either identical or parallel. Take an arbitrary point $\mathbf{p}$ in P_1. If $\mathbf{p}_1$ is in P_2, then P_1 and P_2 are identical. If $\mathbf{p}$ is not in P_2, then P_1 and P_2 are parallel.

If plane P_1 and P_2 are not identical or parallel, then they have a unique intersection line.

Let P_1: $a_1x + b_1y + c_1z = d_1$, P_2: $a_2x + b_2y + c_2z = d_2$, and P_3: $a_3x + b_3y + c_3z = d_3$, be three distinct planes. If any two of them are not parallel, then they may have a unique intersection line or a unique intersection point.

Solve the system of linear equations:

$a_1x + b_1y + c_1z = d_1$,
$a_2x + b_2y + c_2z = d_2$,
$a_3x + b_3y + c_3z = d_3$.

Because P_1, P_2, and P_3 are distinct planes, none of these equations is a multiple of another equation. If this system has no solution, then there is no intersection of these three planes; if this system has infinitely many solutions, then these planes have a unique intersection line; if this system has a unique solution, then these planes have a unique intersection point.

Example 6.2.5. Find the intersection point of the line L: $(x, y, z) = (1, 2, -1) + t\,(2, -1, -1)$, and the plane P: $2x - y + 3z = 1$.

Solution. If L has an intersection with P, then there exists a value of t such that $(x, y, z) =$ $(1\ 2, -1) + t\,(2, -1, -1)$ is in P. Use the parametric form of this lne.

$2(1 + 2t) - (2 - t) + 3(-1 - t) = 1.$

Hence, $2 + 4t - 2 + t - 3 - 3t = 1$, or $-3 + 2t = 1$. $t = 2$.

The intersection point is $(1, 2, -1) + 2(2, -1, -1) = (5, 0, -3)$.

Example 6.2.6. Find the vector equation of the intersection line of two planes P_1: $2x + 3y - 6z =$ 5 and P_2: $x + 2y - 2z = 3$.

Solution. Since the intersection line is in both planes, this line is perpendicular to the normal vector $\mathbf{n}_1$ of P_1 and the normal vector $\mathbf{n}_2$ of P_2. Hence, it is collinear with the cross product $\mathbf{n}_1 \times \mathbf{n}_2$. In other words, $\mathbf{n}_1 \times \mathbf{n}_2$ is a directional vector of the intersection.

Let $\mathbf{n}_1 = (2, 3, -6)$ and $\mathbf{n}_2 = (1, 2, -2)$. Then $\mathbf{n}_1 \times \mathbf{n}_2 = (6, -2, 1)$. We see that $(1, 1, 0)$ is a point that satisfies both equations. This point is on the intersection line. We find a vector form of the intersection line $\mathbf{p} = (1, 1, 0) + t(6, -2, 1)$.

Example 6.2.7. Find the intersection point(s) of the planes $2x - y + 3z = 3$, $x + 2y - z = 4$, $y - z =$ 1.

Solution. Solve the system of equations

$2x - y + 3z = 3, \qquad (1)$

$x + 2y - z = 4,$ (2)
$y - z = 1.$ (3)

(2) × 2 – (1): $5y - 5z = 5$. (4)

This means that equation (3) is actually a linear combination of (1) and (2). Let z be a parameter t. Then $y = 1 + t$, and $x = 4 - 2y + z = 4 - 2(1 + t) + t = 2 - t$. The general solution of this system is

$(x, y, z) = (2 - t, 1 + t, t) = (2, 1, 0) + t(-1, 1, 1)$.

This is the vector form of the intersection line.

Example 6.2.8. Find the intersection point(s) of the planes $2x - y + 3z = 3$, $x + 2y - z = 4$, $y + z = 5$.

Solution. Solve the system of linear equations:

$2x - y + 3z = 3,$ (1)
$x + 2y - z = 4,$ (2)
$y + z = 5.$ (3)

(2) × 2 – (1): $5y - 5z = 5$, $y - z = 1$. (4)
(3) + (4): $2y = 6$, $y = 3$.
Then $z = y - 1 = 2$, and $x = 4 - 2y + z = 4 - 6 + 2 = 0$.

The intersection of these three planes is a point (0, 3, 2).

3. Distance Problems

A. The Distance from a Point to a Line

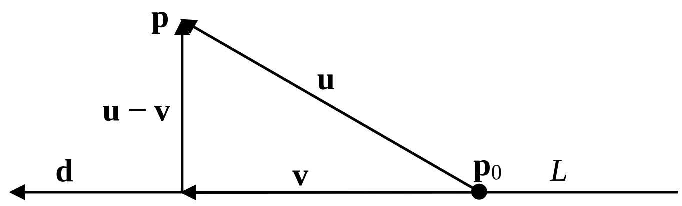
The distance from a point $\mathbf{p}$ to a line L is the shortest distance from $\mathbf{p}$ to a point in L. The distance from $\mathbf{p}$ to L can be found be found by the following steps:

a. Take an arbitrary point $\mathbf{p}_0$ in L.

b. Find the projection of the vector $\mathbf{u} = \mathbf{p} - \mathbf{p}_0$ onto the direction vector $\mathbf{d}$ of L, $\mathbf{v} = \text{proj}_{\mathbf{d}}(\mathbf{p} - \mathbf{p}_0)$.

c. The length of **u** – **v** is the distance from **p** to *L*.

Indeed, what we did is to decompose the vector **u** into two components: one component **v** is in the direction of *L*, and the other **u** − **v** is perpendicular to *L*, and the length of **u** – **v** is the distance from **p** to *L*.



Example 6.3.1. Find the distance from point **p** = (1, −5, 1) to the line *L* : (*x*, *y*, *z*) = (3, −1, −5) + *t* (1, −2, 2).

Solution. Let $\mathbf{p}_0$ = (3, −1, −5). Then **u** = **p** – $\mathbf{p}_0$ = (−2, −4, 6). A directional vector of *L* is **d** = (1, −2, 2). $\mathbf{v} = \text{proj}_{\mathbf{d}}\,\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{d}}{|\mathbf{d}|^2}\mathbf{d} = \dfrac{(-2,-4,6)\cdot(1,-2,2)}{9}(1,-2,2) = (2,-4,4)$.

u – **v** = (−2, −4, 6) – (2, −4, 4) = (−4, 0, 2).

The distance from **p** to *L* is $|\mathbf{u} - \mathrm{v}| = \sqrt{20} = 2\sqrt{5}$.

B. The Distance between Two Parallel Lines of Two Skew Lines

The distance between two parallel lines equals the distance from a point on one line to another line.

Example 6.3.2. Find the distance between lines L_1: (*x*, *y*, *z*) = (3, −1, −5) + *s* (1, −2, 2), and L_2: (*x*, *y*, *z*) = (1, −5, 1) + *t* (−2, 2, −4).

Since the directional vectors of L_1 and L_2 are collinear, these two lines are identical or parallel. Find the distance from a point (1, −5, 1), say, to L_1. The distance is (as found in *Example* 6.3.1) $2\sqrt{5}$. The distance between L_1 and L_2 is $2\sqrt{5}$, and they are no identical.

The distance between two skew lines is defined to be the shortest distance between a point in one line and a point in the other line. This distance is the length of the projection of a line joining one point in one line and a point in another line onto a vector that is perpendicular to both lines.

Let L_1: $\mathbf{p} = \mathbf{p}_1 + t\,\mathbf{v}_1$ and L_2: $\mathbf{p} = \mathbf{p}_2 + t\,\mathbf{v}_2$ be two skew lines. Let $\mathbf{u} = \mathbf{p}_2 - \mathbf{p}_1$ be the vector from $\mathbf{p}_1$ to $\mathbf{p}_2$. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then $|\,\mathrm{proj}_{\mathbf{n}}\,\mathbf{u}\,|$ is the distance between L_1 and L_2.

Example 6.3.3.Consider lines L_1: $(x, y, z) = (0, -1, 1) + s\,(1, 2, -1)$, and L_2: $(x, y, z) = (0, -2, 3) + t\,(2, 1, 1)$.

a. Show these two lines are skew lines.
b. Find the distance between these two lines.

Solution. a. If L_1 and L_2 have an intersection, then we have values s and t satisfying the system

$s = 2t$, (1)
$2s - 1 = t - 2$, (2)
$1 - s = t + 3$. (3)

Plug $s = 2t$ in to (2) and (3).

$4t - 1 = t - 2$, (4)
$1 - 2t = t + 3$. (5)

From (4), we have $t = -1 / 3$. From (5) we have $t = -2 / 3$. Hence, this system does not have a solution, and L_1 and L_2 are skew lines.

b. Let $\mathbf{p}_1 = (0, -1, 1)$ and $\mathbf{p}_2 = (0, -2, 3)$. Then $\mathbf{u} = \mathbf{p}_2 - \mathbf{p}_1 = (0, -1, 2)$.

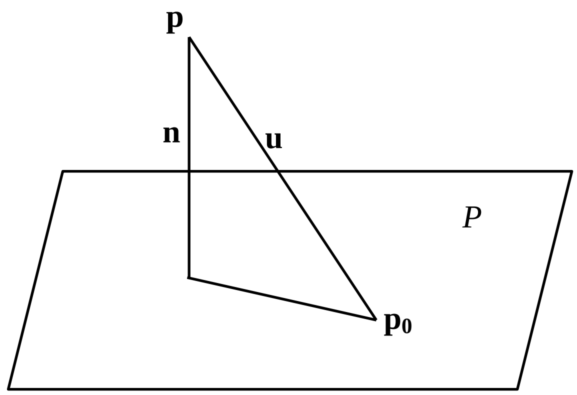
Let $\mathbf{v}_1 = (1, 2\ {-1})$, and $\mathbf{v}_2 = (2, 1, 1)$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (3, -3, -3)$.

The distance between these two lines is

$$d = |\,\mathrm{proj}_{\mathbf{n}}\,\mathbf{u}\,| = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{|\,\mathbf{n}\,|}\right| = \left|\frac{(0,-1,2)\cdot(3,-3,-3)}{3\sqrt{3}}\right| = \frac{1}{\sqrt{3}}.$$

C. The Distance from a Point to a Plane, between a Line and a Parallel Plane, or Two Parallel Planes

Idea. Let $\mathbf{p}_0$ be a point on the plane. Let vector $\mathbf{u} = \mathbf{p} - \mathbf{p}_0$. Then the length of the projection of $\mathbf{u}$ onto a normal vector $\mathbf{n}$ is the distance from $\mathbf{p}$ to P.



Example 6.3.4. Find the distance from a point **p** = (3, 2, 0) to plane *P*: $8x - y - 4z = 4$.

Solution. It is easy to find that a point $\mathbf{p}_0 = (0, 0, -1)$ is on *P*. Vector **n** = (8, –1, –4) is a normal vector of *P*.

Let $\mathbf{u} = \mathbf{p} - \mathbf{p}_0 = (3, 2, 1)$. The distance from **p** to *P* is the length of the projection of **u** onto **n**:

$$|\operatorname{proj}_{\mathbf{n}} \mathbf{u}| = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{n}|}\right| = \frac{18}{9} = 2.$$

The distance between a line *L* and a plane *P* parallel to *L* is the distance from a point **p** in *L* to *P*.

The distance between two parallel planes P_1 and P_2 is the distance from a point **p** in P_1 to P_2.

Example 6.3.5. Find the distance between two parallel planes P_1: $x - 2y + 2z = 3$ and P_2: $-2x + 4y - 4z = 10$.

Solution. These two planes are parallel because their normal vectors are collinear. P_1 and P_2 are not identical because point **p** = (1, 0, 1) is on P_1 but it is not on P_2.

The distance between P_1 and P_2 is the distance from **p** to P_2. A normal vector of P_2 is **n** = (−2, 4, −4). Take a point $\mathbf{p}_0 = (-1, 1, -1)$ on P_2. Then $\mathbf{u} = \mathbf{p} - \mathbf{p}_0 = (2, -1, 2)$.

$$|\operatorname{proj}_{\mathbf{n}} \mathbf{u}| = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{n}|}\right| = \frac{16}{6} = \frac{8}{3}.$$

The distance between P_1 and P_2 is $\frac{8}{3}$.