TOPIC 3. SINUSOIDAL FUNCTIONS AND THEIR DERIVATIVES

1. SINUSOIDAL FUNCTIONS

Read: § 4.1.

A. Sinusoidal Functions

A sinusoidal function has the form $y = A\sin(B(t+\phi)) + M$, where A is called the *amplitude*, $F = \frac{B}{2\pi}$ is called the *frequency*, ϕ is called the *phase*, and M is called the *mean*. The variable of a sinusoidal function is usually the time t.

Since $-1 \le \sin t \le 1$, the sinusoidal function takes values between $M - A$ and $M + A$. The frequency give the number of waves in every unit of time, and the phase determines the behaviour of the function at time $t = 0$. If the time is measured by seconds, the frequency gives the number of waves in each second, and the unit is called *hertz*.

Since $y\left(t+\frac{2\pi}{B}\right) = A\sin\left(B\left(t+\frac{2\pi}{B}+\phi\right)\right)+M = A\sin\left(B(t+\phi)+2\pi\right)+M = y(t)$, the value $T = \frac{2\pi}{B}$ is called the (*smallest*) *period* of this function.

Example 3.1.1. Find the amplitude, mean, frequency, smallest period, and phase of the function

$y = 2\sin\left(3t-\frac{\pi}{4}\right) - 1.$

Solution. Write this function in the standard form. $y = 2\sin\left(3\left(t-\frac{\pi}{12}\right)\right) - 1$. The amplitude is $A = 2$, the mean is $M = -1$, the phase is $\phi = -\frac{\pi}{12}$, the frequency is $F = \frac{3}{2\pi}$, and the smallest period is $T = \frac{2\pi}{3}$.

Sinusoidal functions are frequent used to form periodic models.

B. Graphs of Sinusoidal Functions

Recall the following facts about the graphs of functions:

(i) The graph of the function $y = f(x + a)$, $a > 0$, is obtained from the graph of $y = f(x)$ by horizontally shift to the left for a distance a.

(ii) The graph of the function $y = f(x - a)$, $a > 0$, is obtained from the graph of $y = f(x)$ by horizontally shift to the right for a distance a.

(iii) The graph of the function $y = f(x) + a$, $a > 0$, is obtained from the graph of $y = f(x)$ by vertically shift up for a distance a.

(iv) The graph of the function $y = f(x) - a$, $a > 0$, is obtained from the graph of $y = f(x)$ by vertically shift down for a distance a.

(v) The graph of the function $y = f(kx)$, $k > 1$, is obtained from the graph of $y = f(x)$ by compressing it horizontally for a factor k.

(vi) The graph of the function $y = f(x / k)$, $k > 1$, is obtained from the graph of $y = f(x)$ by stretching it horizontally for a factor k.

(vii) The graph of the function $y = kf(x)$, $k > 1$, is obtained from the graph of $y = f(x)$ by stretching it vertically for a factor k.

(vi) The graph of the function $y = f(x) / k$, $k > 1$, is obtained from the graph of $y = f(x)$ by compressing it vertically for a factor k.

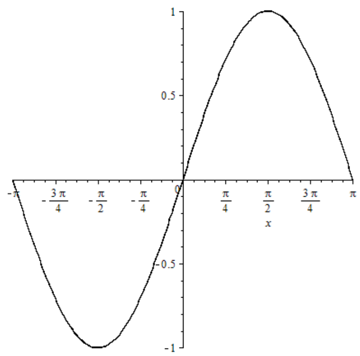
(ix) The graph of the function $y = f(-x)$ is the mirror image of the graph of $y = f(x)$ with respect to the y-axis.

(x) The graph of the function $y = -f(x)$ is the mirror image of the graph of $y = f(x)$ with respect to the x-axis.

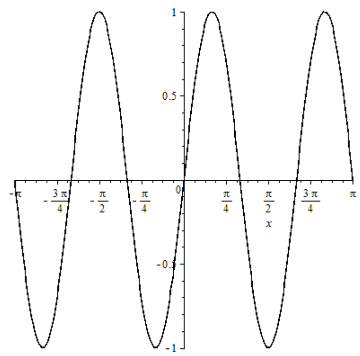
The graph of a sinusoidal function is obtained from the graph of the function $y = \sin t$ with a number of these operations.

Example 3.1.2. Find the graph of the function $y = 2\sin\left(3t - \frac{\pi}{4}\right) - 1$.

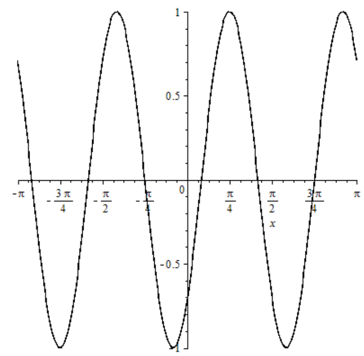
Solution. Use the standard form $y = 2\sin\left(3\left(t - \frac{\pi}{12}\right)\right) - 1$. The graph of this function is obtained in the following steps:



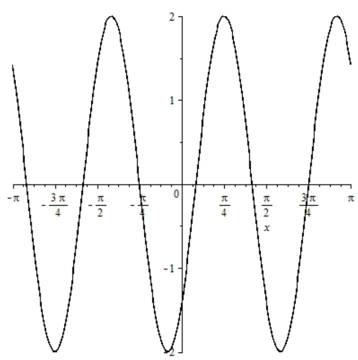
$y = \sin t$



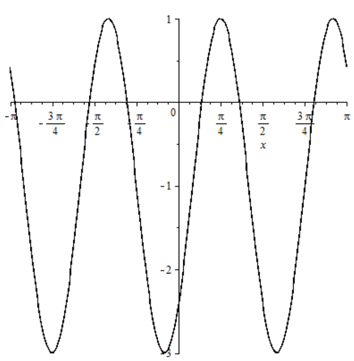
$y = \sin(3t)$



$y = \sin\left(3t - \frac{\pi}{12}\right)$



$y = 2\sin\left(3t - \frac{\pi}{12}\right)$



$y = 2\sin\left(3t - \frac{\pi}{12}\right) - 1$

Because $\sin x = \cos\ = \cos\left(x + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right)$. A sinusoidal function may also be expressed by the cosine function. For instance, $2\sin\left(3t - \frac{\pi}{12}\right) - 1 = 2\cos\left(3t - \frac{\pi}{12} + \frac{\pi}{2}\right) - 1 = 2\cos\left(3t + \frac{5\pi}{12}\right)$.

2. Derivatives of the Sine and Cosine Functions

Formulas:

$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

Example 3.2.1. Find the derivative of the following functions:

a. $y = 3\sin x + 4\cos x$.

b. $y = x^2 \sin(3x)$.

c. $y = \cos(x^2)$.

d. $y = \sin^3 x$.

e. $y = \tan x$.

f. $y = \text{scs}\, x$.

Solution. a. Use the sum rule. $y' = 3\cos x - 4\sin x$.

b. Use the product rule and the chain rule. $y' = 2x \sin(3x) + 3x^2 \cos(3x)$.

c. Use the chain rule. $y' = -2x \cos(x^2)$.

d. Use the chain rule. $y' = 3\sin^2 x \cos x$.

e. $y = \dfrac{\sin x}{\cos x}$. Use the quotient rule. $y' = \dfrac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$.

f. $y = \dfrac{1}{\cos x} = (\cos x)^{-1}$. Use the chain rule and the power rule. $y' = -\dfrac{1}{\cos^2 x}(-\sin x) = \dfrac{\sin x}{\cos^2 x}$
$= \tan x \sec x$.

Example 3.2.2. Find the equation of the tangent line of the function $y = \cos(3x) - \sin x$ at $x = \dfrac{\pi}{6}$.

$$y\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{2} - \sin\frac{\pi}{6} = -\frac{1}{2}.$$

$y' = -3\sin(3x) - \cos x$. $y'\left(\dfrac{\pi}{6}\right) = -3\sin\dfrac{\pi}{2} - \cos\dfrac{\pi}{6} = -3 - \dfrac{\sqrt{3}}{2}$.

The equation of the tangent line is $y = -\left(3+\frac{\sqrt{3}}{2}\right)\left(x-\frac{\pi}{6}\right)-\frac{1}{2}$.

3. Applications of Sinusoidal Functions

A. Alternating Current (AC)

The voltage of an alternating current electric circuit changes according to a sinusoidal function with a zero mean. Choose an appropriate origin of time, we can write

$V(t) = A \sin(\omega t)$.

Then the voltage is between $-A$ and A. The frequency is $F = \frac{\omega}{2\pi}$, and the period is $T = \frac{2\pi}{\omega}$.

If the circuit has a direct current component (DC), then the voltage is expressed by

$V(t) = A \sin(\omega t) + M$, where M is the direct current component.

The voltage is between $M - A$ and $M + A$.

Example 3.3.1. If the voltage of an electric circuit is given by $V(t) = 110\sin(120\pi t)+12$, where t is the time in seconds. Find the maximum, minimum of the voltage, and the time at which they occur, and the frequency.

Solution. The maximum voltage is 110 + 12 = 122, and the minimum voltage is −110 + 12 = −98. The maximum voltage occurs when $\sin(110\pi t) = 1$. Since the $\sin x = 1$ when $x = \left(2n+\frac{1}{2}\right)\pi$, where n is an integer, we have $110\pi t = \left(2n+\frac{1}{2}\right)\pi$. The maximum voltage occurs at time $t = \frac{1}{110}\left(2n+\frac{1}{2}\right)\pi = \frac{1}{220}(4n+1)\pi$, where n is an integer. Similarly, since $\sin x = -1$ when $x = \left(2n-\frac{1}{2}\right)\pi$, and the minimum voltage occurs at time $\frac{1}{220}(4n-1)\pi$, where n is an integer.

The frequency is $F = \dfrac{120\pi}{2\pi} = 60$ hertz.

B. Simple Harmonic Motion

If the position of a particle moving along the *x*-axis is given by a sinusoidal function, we say that this particle has a *harmonic motion*. We may express the position function of a harmonic motion by

$$x(t) = A\sin(\omega(t+\phi)) + x_0$$

Then this particle moves between $x = x_0 + A$ and $x = x_0 - A$. The phase ϕ depends on the choice of the origin of time. If let $t = 0$ be the time when the particle is at the center $x = x_0$, we may let $\phi = 0$. Then $x(0) = A\sin(\omega 0) + x_0 = A\sin 0 + x_0 = x_0$. If let $t = 0$ be the time when the particle has the maximum displacement $x = x_0 + A$, we may let $\phi = \dfrac{\pi}{2\omega}$. Then $x(0) = A\sin\left(\omega\left(0 + \dfrac{\pi}{2\omega}\right)\right) + x_0 = A\sin\dfrac{\pi}{2\omega} + x_0 = x_0 + A$. The frequency is $F = \dfrac{\omega}{2\pi}$, and the period is $T = \dfrac{2\pi}{\omega}$.

The velocity of the particle is given by the derivative

$$x'(t) = A\omega\cos(\omega(t+\phi)).$$

The acceleration of the particle is given by the second derivative

$$x''(t) = -A\omega^2\sin(\omega(t+\phi)).$$

An example of a harmonic motion is the horizontal position of the bob of a simple pendulum. If the length of a pendulum is L, in meters, then the period, in seconds, is given by

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the acceleration of the gravity. Since $T = \dfrac{2\pi}{\omega}$, $\omega = \dfrac{2\pi}{T} = \sqrt{\dfrac{g}{L}}$. The horizontal position of the bob of the pendulum is given by

$x(t) = A\sin\left(\frac{2\pi}{T}(t+\phi)\right) = A\sin\left(\sqrt{\frac{g}{L}}(t+\phi)\right)$,

where A is the maximum horizontal displacement of the bob.

Example 3.3.2. Suppose a pendulum has length 1 meter, and the maximum displacement is 0.1 meter. Choose time $t = 0$ when the bob has the maximum displacement $x = 0.1$. Find the times when the bob of the pendulum has the maximum displacement, the maximum speed and the maximum acceleration.

Solution. Since $L = 1$, and $A = 0.1$, $x(t) = 0.1\sin\left(\sqrt{g}t+\phi\right)$.

When $t = 0$, we want $x(0) = 0.1 = 0.1\sin\phi$. Then $\sin\phi = 1$. Choose $\phi = \frac{\pi}{2}$.

Then $x(t) = 0.1\sin\left(\sqrt{g}t+\frac{\pi}{2}\right)$,

$x'(t) = 0.1\sqrt{g}\cos\left(\sqrt{g}t+\frac{\pi}{2}\right)$, and

$x''(t) = -0.1g\sin\left(\sqrt{g}t+\frac{\pi}{2}\right)$.

The maximum displacement and the maximum acceleration occur at sin $\sin\left(\sqrt{g}t+\frac{\pi}{2}\right) = \pm 1$, $\sqrt{g}\,t = n\pi$, or $t = \frac{n\pi}{\sqrt{g}}$, where n is an integer. The maximum speed occurs at $\cos\left(\sqrt{g}t+\frac{\pi}{2}\right) = \pm 1$, $\sqrt{g}\,t = \left(n+\frac{1}{2}\right)\pi$, or $t = \frac{\pi}{\sqrt{g}}\left(n+\frac{1}{2}\right)$, where n is an integer.

TOPIC 4. EXPONENTIAL FUNCTIONS AND LOGARITHMIC FUNCTIONS

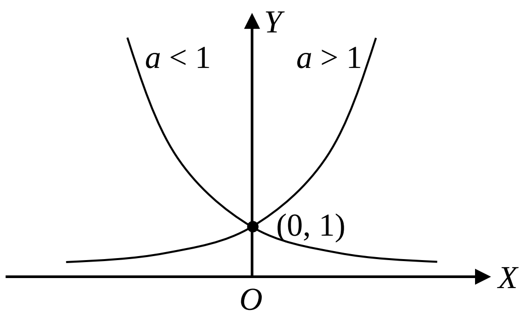
Read: §§ 5.1-5.5

1. THE EXPONENTIAL FUNCTION

A. Exponential Functions

Exponential functions have the general form $y = Ca^{kx}$, where C, k and a are constants, $a > 0$ and $a \neq 1$.

The graph of $y = a^x$ is illustrated in the following figure:



The graph of the general exponential function $y = Ca^{kx}$ can be obtained from the graph of the function $y = a^x$ by vertical / horizontal stretch / compression.

B. Properties of the Exponential Function

From the graph, we can "read" a number of important properties of this function:

(i) Domain: $-\infty < x < \infty$.

(ii) Range: $y > 0$.

(iii) Monotonicity: When $a > 1$, the function is increasing; when $a < 1$, this function is decreasing.

(iv) Concavity: Always concave up.

(v) Continuity: The function is continuous for all x.

(vi) $f(0) = 1$.

(vii) When $a > 1$, $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to-\infty} a^x = 0$; when $a < 1$, $\lim_{x\to\infty} a^x = 0$, $\lim_{x\to-\infty} a^x = \infty$. Hence, in either case, $y = 0$ is a horizontal asymptote.

We may also need the following fact in some applications: When $a > 1$, exponential function increases faster than any power function, namely, when $a > 1$,

$\lim_{x\to\infty} \frac{x^k}{a^x} = 0$ for any k.

C. The Natural Exponential Function

If an irrational number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$ is used as the base of the exponential function, then this exponential function is called the *natural exponential function*.

The reason of using e as the base is that the limit $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$, which makes many calculations simpler.

2. LOGARITHMIC FUNCTIONS

A. The Logarithm

If $a^b = c$, where $a > 0$ and $a \neq 1$, the *logarithm* of c with *base a* is b, denoted by $b = \log_a c$. In words, the logarithm of c with base a is the **exponent** when c is expressed as a power of a.

If e is used as the base, the logarithm is called the *natural logarithm*. We use symbol $\ln x$ to denote $\log_e x$.

Properties of Logarithm

(a) $\log_a 1 = 0$. (In particular, $\ln 1 = 0$).

(b) $\log_a a = 1$. (In particular, $\ln e = 1$).

Assume $x > 0$, $y > 0$.

(c) $\log_a (xy) = \log_a x + \log_a y$.

(d) $\log_a (x / y) = \log_a x - \log_a y$.

(e) $\log_a x^k = k \log_a x$.

(f) $a^{\log_a x} = x$. (In particular, $e^{\ln x} = x$).

(g) $\log_a x = \dfrac{\log_b x}{\log_b a}$. (The base changing formula)

In particular, when $b = e$, $\log_a x = \dfrac{\ln x}{\ln a}$.

With the properties of the logarithm, we can solve equations with exponents and / or logarithm.

Example 4.2.1. Solve $\log_2 (x + 1) - \log_2 (2x - 3) = 2$.

$\log_2 \dfrac{x+1}{2x-3} = 2, \dfrac{x+1}{2x-3} = 4$, $8x - 12 = x + 1$, $7x = 13$, $x = \dfrac{13}{7}$.

In the first step, we used the formula $\log_a x + \log_a y = \log_a (xy)$. But this formula requires that $x > 0$ and $y > 0$. Therefore, when we get the value of x, we have to verify the validity of the root. Since $\dfrac{13}{7} + 1 > 0$ and $2 \times \dfrac{13}{7} - 3 > 0$, the root is valid.

Example 4.2.2. Solve $2^{2x+1} = 3^{x-1}$.

$(2x + 1) \ln 2 = (x - 1) \ln 3$, $(2 \ln 2)x + \ln 2 = (\ln 3)x - \ln 3$, $(2 \ln 2 - \ln 3)x = -\ln 2 - \ln 3$.

$x = -\dfrac{\ln 2 + \ln 3}{2\ln 2 - \ln 3}$.

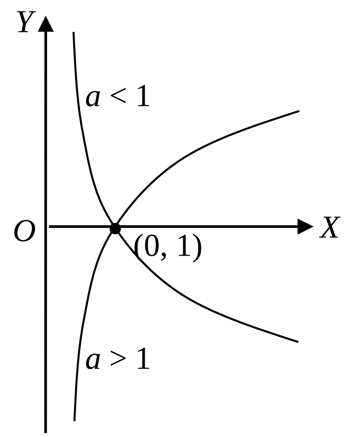
Example 4.2.3. Solve $e^{2x} - 2e^x + 3 = 0$.

Let $t = e^x$. Then $t^2 - 2t + 3 = 0$, $t = -1, 3$. Since $t = e^x > 0$, $t = -1$ is invalid. Then t = 3, and $x = \ln 3$.

B. The Logarithmic Function

The *logarithmic function* $y = \log_a x$, $a > 0$ and $a \neq 1$, is the inverse of the exponential function $y = a^x$.

The graph of the logarithmic function is illustrated in the following figure:



C. Properties of the Logarithmic Function

From the graph, we can find a number of important properties of the logarithmic function $y = f(x)$, where $f(x) = \log_a x$.

(a) Domain: $x > 0$.

(b) Range: $-\infty < y < \infty$.

(c) Monotonicity: When $a < 1$, the function is decreasing; when $a > 1$, this function is increasing.

(d) Concavity: When $a < 1$, the graph is concave up; when $a > 1$, the graph is concave down.

(e) Continuity: The function is continuous for $x > 0$.

(f) $f(1) = 0$.

(g) When $a > 1$, $\lim_{x\to\infty}\log_a x = \infty$, $\lim_{x\to 0^+}\log_a x = -\infty$; when $a < 1$, $\lim_{x\to\infty}\log_a x = 0$, $\lim_{x\to 0^+}\log_a x = \infty$. Hence, in either case, $x = 0$ is a vertical asymptote.

We may also need the following fact in some applications: Although, when $a > 1$, logarithmic function approaches infinity when x approaches infinity, but it increases slower than any power function $y = x^k$, where $k > 0$. Namely, when $a > 1$,

$$\lim_{x\to\infty}\frac{\log_a x}{x^k} = 0 \text{ for any } k > 0.$$

D. Expressing a General Exponential / Logarithmic Function as a Natural Exponential / Logarithmic Function

Since $a = e^{\ln a}$, $a^x = e^{(\ln a)x}$. Then we can always express a general exponential function $y = a^x$ as a natural exponential function $y = e^{(\ln a)x}$.

Example 4.2.4. $2^x = e^{(\ln 2)x}$.

By the base changing formula, we can express a general logarithmic function $y = \log_a x$ as a natural logarithmic function $y = \dfrac{\ln x}{\ln a}$.

Example 4.2.5. $\log_2 x = \left(\dfrac{1}{\ln 2}\right)\ln x$.

3. Exponential Models

A. Compound Interest

Suppose that an amount A_0 is deposited to a bank account with annual interest rate r, compounded n times a year.

Denote the balance of the account after t years by $A(t)$. We want to find an expression to calculate $A(t)$.

The interest earned after $\dfrac{1}{n}$ year is $\dfrac{r}{n}A_0$. The balance after $\dfrac{1}{n}$ year is $A\left(\dfrac{1}{n}\right) = A_0 + \dfrac{r}{n}A_0$

$= A_0\left(1+\frac{r}{n}\right)$.

The interest earned between $\frac{1}{n}$ year and $\frac{2}{n}$ year is $\frac{r}{n}A\left(\frac{1}{n}\right)$. The balance after $\frac{2}{n}$ year is

$A\left(\frac{2}{n}\right) = A\left(\frac{1}{n}\right)\left(1+\frac{r}{n}\right) = A_0\left(1+\frac{r}{n}\right)^2$.

In general, the balance after k / n years is $A\left(\frac{k}{n}\right) = A_0\left(1+\frac{r}{n}\right)^k$.

If we write $t = \frac{k}{n}$, then $k = nt$.

The balance after t years is $A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$.

This is an exponential function.

Example 4.3.1. Suppose $15,000 is deposited into a bank account with annual interest rate 3% compounded monthly.

(a) Find the balance after 5 year.

(b) How long would it take for the balance to reach $20,000?

Solution. (a) The interest rate for every month is 0.03 / 12 = 0.0025. There are 12 × 5 = 60 months in 5 years. The balance after 5 years is

$A(5) = 15000 \times 1.0025^{60} = 17424.25$.

(b) Assume that the balance will reach $20,000 after T years. Then

$A(T) = 20000 = 15000 \times 1.0025^{12t}$. $1.0025^{12t} = 20000 / 15000 = 4 / 3$.

$12T = \log_{1.0025} (4 / 3) = \ln (4 / 3) / \ln 1.0025 \approx 115.22$. $T \approx 115.22 / 12 \approx 9.6$.

After 9.6 years, i.e., approximately 9 years and 7 months, the balance will reach $20,000.

When n approaches infinity, the limit of function $A(t)$ is

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = \lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}(rt)} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}}\right)^{rt} = \left(\lim_{n/r\to\infty}\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}}\right)^{rt} = e^{rt}.$$

$A(t) = A_0 r^{rt}$ is said to be the balance of the account if *the interest is compounded continuously*.

Example 4.3 2. If the interest is compounded continuously with annual interest rate r%, how long would it take to double the deposit?

If $A(T) = 2A_0 = A_0 e^{\frac{r}{100}T}$, then $e^{\frac{r}{100}T} = 2$, $\frac{r}{100}T = \ln 2$, $T = \frac{100\ln 2}{r}$. Since $\ln 2 \approx 0.7$. $T \approx 70 / r$.

In economics, this is so-called the "70% rule". For example, if the interest rate is $r = 3.5\%$, then it would take approximately 70 / 3.5 = 20 years to double the deposit; if the interest rate is 7%, then it would take approximately 70 / 7 = 10 years to double the deposit.

B. Population Growth

As a first order approximation, the population growth can be studied by the *exponential model*.

Suppose the population of a country (or a region, a city), or the population of a culture of bacteria, is P_0 at time $t = 0$. Then the population after t units of time (years, minutes, seconds, and so on) is $P(t) = P_0(1 + r)^t$, where $r = \frac{P(1) - P_0}{P_0}$ is the *relative rate of growth* of every unit of time.

Example 4.3.3. According to UN, the current world population is 7,500,000,000, and the relative annual rate of growth is 1.11%. If the population keeps growing according to the exponential model with this relative rate, what is the world population at the end of this century?

Solution. Let $t = 0$ be the year of 2017. Then, at 2099, $t = 2099 - 2017 = 82$. Use billion as the unit of the population.

$P(82) = 7.5 \times 1.0111^{82} \approx 18.54$.

The population in 2099 will be approximately 18.54 billion.

Example 4.3.4. Suppose the population of a culture of bacteria grows according to the exponential model. The population is 2000 at 8:00, and the population grows to 3000 at 10:00. What is the population at 14:00?

Solution. Instead of years, in this example, we use hours as the unit of time. Time $t = 0$ is set at 8:00. Then $P_0 = 2000$, and the population at 10:00 is

$P(2) = 3000 = 2000(1 + r)^2$.

Hence, $(1 + r)^2 = \dfrac{3000}{2000} = 1.5$, and $1 + r = 1.5^{1/2}$. The model is

$P(t) = 2000 \times 1.5^{t\ /\ 2}$.

The population at 14:00 is $P(6) = 2000(1 + r)^6 = 2000 \times 1.5^3 = 6750$.

4. The Derivatives of Exponential and Logarithmic Functions

Formulas:

$(e^x)' = e^x$,

$(\ln x)' = 1 / x$.

Use these formulas, we can find the derivatives of the general exponential / logarithmic functions:

Since $a^x = e^{(\ln a)x}$, by the chain rule, we have

$(a^x)' = (\ln a)a^x$.

Since $\log_a x = \dfrac{\ln x}{\ln a}$,

$(\log_a x)' = \dfrac{1}{(\ln a)x}$.

Example 4.3.5. Find the derivative of the following functions:

(a) e^{-2x}.

(b) xe^{3x}.

(c) $\dfrac{\ln x}{x}$.

(d) $2^{\sin x}$.

(e) 1.2^{-x}.

Solution. (a) Use the chain rule, $(e^{-2x})' = -2e^{-2x}$.

(b) By the product rule, $(xe^{3x})' = e^{3x} + 3xe^{3x} = (1 + 3x)e^{3x}$.

(c) By the quotient rule, $\left(\dfrac{\ln x}{x}\right)' = \dfrac{\frac{1}{x}x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$.

(d) By the chain rule, $(2^{\sin x})' = (\ln 2)\, 2^{\sin x}(\cos x)$.

(e) By the chain rule, $(1.2^{-x}) = -\dfrac{1}{\ln 1.2} 1.2^{-x}$.

Example 4.3.6. Suppose the population of a culture of bacteria grows according to the exponential model. The population is 2000 at 8:00, and the population grows to 3000 at 10:00, How fast is the population of the bacteria increasing at 10:00 (given by the increase of the population every minute)?

Solution. We established the model $P(t) = 2000 \times 1.5^{t/2}$ in Example 4.3.4.

Use the chain rule, $P'(t) = 2000 \times \dfrac{1}{2} \ln 1.5 \times 1.5^{t/2}$.

At 10:00, $t = 6$. $P'(6) = 1000 \times \ln 1.5 \times 1.5^3 = 6750 \times \ln 1.5 \approx 1368$.

Because we are using hour as the unit of time, this means that the rate of increasing of the population is about 1368 per hour, or 22.8 per minute. At 10:01, we would expect to have a population 3000 + 23 = 3023.

Example 4.3.7. For which value(s) does the graph of the function $f(x) = x \ln x$ have a horizontal tangent line?

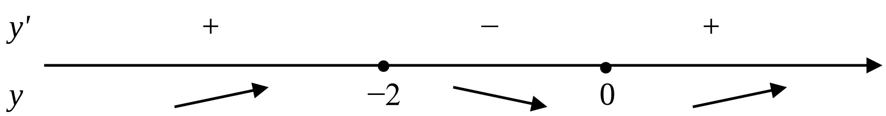
Solution. Use the product rule. $f'(x) = \ln x + 1$. The graph of this function has a horizontal tangent line if and only if the derivative is zero. Let $f'(x) = 0$. $\ln x = -1$, $x = e^{-1}$. When $x = e^{-1}$, the graph of the function $f(x)$ has a horizontal tangent line.

Example 4.3.8. Use the first / second derivative analysis to study the properties of the function $y = x^2e^x$, and sketch the graph of this function.

Solution. *First derivative analysis*:

$y' = 2xe^x + x^2e^x = x(2 + x)e^x$. Let $y' = 0$. $x = 0$, $x = -2$.

Critical numbers $x = 0, -2$. These critical numbers subdivide the domain of this function into intervals $x < -2$, $-2 < x < 0$, and $x > 0$. The signs of the derivative in each of these intervals are shown in the following diagram:



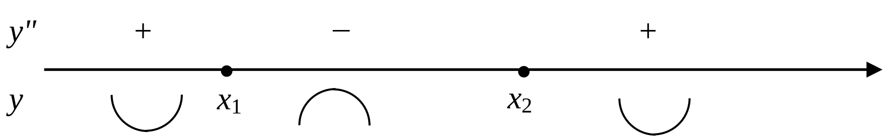
This function increases when $x < -2$ or $x > 0$, and it decreases when $-2 < x < 0$. The function has a local maximum $y(-2) = 4e^2 \approx 0.54$, and it has a local minimum $y(0) = 0$.

Since $\lim_{x\to\infty} y = \infty$, this function does not have the global maximum. Since $y > 0$ when $x \neq 0$, the local minimum $y(0) = 0$ is the global minimum.

Second derivative analysis:

$y'' = (2 + 4x + x^2)e^x$. Let $y'' = 0$. $x = x_1 = -2 - \sqrt{2} \approx -3.41$, and $x = x_2 = -2 + \sqrt{2} \approx -0.59$.

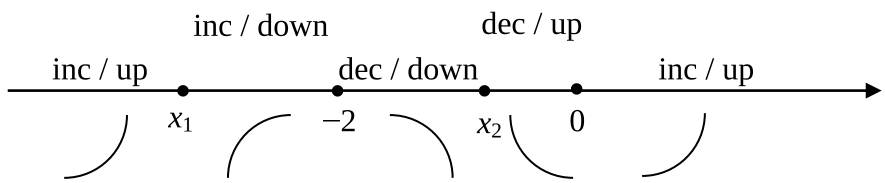
The signs of the second derivative are as in the following diagram:



The graph of this function is concave up when $x < x_1$ or $x > x_2$, and it is concave down when

$x_1 < x < x_2$. It has inflection points at $x = x_1$ and $x = x_2$. $y(x_1) \approx 0.38$. $y(x_2) \approx 0.19$.

The shape of the graph in each interval is shown in the following diagram:



Since this function is defined and continuous for all real numbers x, it does not have a vertical asymptote. Since $\lim_{x\to-\infty} y = 0$, $y = 0$ is a horizontal asymptote.

The graph of this function is the following: