TOPIC 2. CURVE SKETCHING

1. First Derivative Analysis

Read: §§ 3.1, 3.2.

A. Increasing / Decreasing Functions

If $f'(x) > 0$ at a point $x = a$, then function $f(x)$ "increases at $x = a$". This means that function $f(x)$ increases in an interval around the point $x = a$. If $f'(x) < 0$ at a point $x = a$, then function $f(x)$ "decreases at $x = a$". This means that function $f(x)$ decreases in an interval around the point $x = a$.

To determine the intervals where function $y = f(x)$ increases / decreases, we need to know the intervals where $f'(x)$ is positive / negative.

The domain of a function consists of a number of intervals. Suppose function $y = f(x)$ is defined and has a continuous derivative in an interval (a, b). Let c be a point in this interval, $a < c < b$. If $f'(x)$ changes its sign when x goes from one side of c to the other side of c, then we must have one of the following two cases:

a. $f'(c) = 0$.

b. $f'(c)$ is not defined.

This observation leads to an important definition:

A number $x = c$ is called a *critical number* of a function $y = f(x)$ if

a. Value c is in the domain of $f(x)$, (in other words, $f(c)$ is defined), and

b. The derivative $f'(c) = 0$, or $f'(x)$ does not exist.

Critical numbers subdivide the domain of $f(x)$ into a number of subintervals. In each of these subintervals, the derivative $f'(x)$ does not change its sign. Then we may choose an arbitrary value in each subinterval and plug it into $f'(x)$ to find out the sign of $f'(x)$ in this subinterval.

Use Critical Numbers to Determine the Intervals where Function y = f (x) Increases / Decreases

a. Find the derivative of $f(x)$, and find the critical numbers.

b. The critical numbers subdivide the domain of $f(x)$ into a number of subintervals.

c. Choose a particular value c in each of the subintervals. Plug this number into the first derivative $f'(x)$. The signs of $f'(x)$ for all x in this subinterval are the same as $f'(c)$.

d. Use the sign of $f'(x)$ to determine the interval(s) where $f(x)$ is increasing / decreasing.

Example 2.1.1. Consider function $y = x^3 + 2x^2 - 4x + 2$, $-4 \le x \le 1$. Find the interval(s) where this function is increasing / decreasing.

Solution. The derivative is $y' = 3x^2 + 4x - 4 = (x + 2)(3x - 2)$. The derivative is defined everywhere in the domain of this function, so we don't have a value of x where y' is not defined. To find the critical numbers, let $y' = (x + 2)(3x - 2) = 0$. We have critical numbers $x = -2, \frac{2}{3}$. The domain of this function is the interval $[-4, 1]$. These two critical numbers subdivide the domain of the function into three subintervals:

$(-4, -2)$, $\left(-2, \frac{2}{3}\right)$, and $\left(\frac{2}{3}, 1\right)$.

In the first interval, choose $x = -3$. Since $y'(-3) = 11 > 0$, $y' > 0$ in $(-4, -2)$. In the second interval, choose $x = 0$. Since $y'(0) = -4 < 0$, $y' < 0$ in $\left(-2, \frac{2}{3}\right)$. In the third interval, let x be a value less than but close to 1, then y' is close to $(1 + 3)(3 - 1) = 4 \times 1 = 4 > 0$. Hence, $y' > 0$ in $\left(\frac{2}{3}, 1\right)$.

Therefore, this function increases in intervals $(-4, -2)$ and $\left(\frac{2}{3}, 1\right)$, and it decrease in interval $\left(-2, \frac{2}{3}\right)$.

B. Local Maxima / Minima

Local extrema: If there exists an interval (a, b) in the domain of a function $y = f(x)$ and $a < c < b$ such that $f(c) \ge f(x)$ for every value x in (a, b), then this function attains a *local* (or *relative*) *maximum* $f(c)$ at $x = c$; if $f(c) \le f(x)$ for every value x in (a, b), then this function attains a *local* (or *relative*) *minimum* $f(c)$ at $x = c$. A *local* (or *relative*) *extremum* is either a local maximum or a local minimum.

If $y = f(x)$ is continuous in an interval (a, b) around the value $x = c$, $f(x)$ attains a local maximum at $x = c$ if and only if this function is increasing when $a < x < c$, and it is decreasing when $c < x < b$, $f(x)$ attains a local minimum at $x = c$ if and only if this function is decreasing when $a < x < c$, and it is increasing when $c < x < b$. This means that, in either case, $f'(x)$ changes its sign when x goes from one side of c to the other side of c. In other words, $x = c$ is a critical number.

If function $f(x)$ attains a local extremum at $x = c$, then $x = c$ is a critical number.

The converse is not necessarily true. If $x = c$ is a critical number of $f(x)$, function $f(x)$ may not attain a local extremum at $x = c$ because $f'(x)$ may have the same sign on both sides of c.

First Derivative Test:

Let $f(x)$ be a function defined and has a continuous derivative in an interval (a, b). Suppose c is a critical number of $f(x)$, $a < c < b$. If $f'(x) > 0$ when $a < x < c$, and $f'(x) < 0$ when $c < x < b$, then $f(x)$ attains a local maximum at $x = c$. If $f'(x) < 0$ when $a < x < c$, and $f'(x) > 0$ when $c < x < b$, then $f(x)$ attains a local minimum at $x = c$.

Exmaple 2.1.1. (Continued) Consider function $y = x^3 + 2x^2 - 4x + 2$, $-4 \le x \le 1$.

Since this function increases in intervals $(-4, -2)$ and $\left(\frac{2}{3}, 1\right)$, and it decrease in interval $\left(-2, \frac{2}{3}\right)$, it attains a local maximum at $x = -2$, $y(-2) = 10$, and it attains a local minimum at $x = \frac{2}{3}$, $y\left(\frac{2}{3}\right) = \frac{14}{27} \approx 0.52$.

C. Absolute Maximum / Minimum

If, for every x in the domain of this function $f(c) \ge f(x)$, then this function attains an *absolute* (or *global*) *maximum* $f(c)$ at $x = c$. If, for every x in the domain of this function $f(c) \le f(x)$, then this function attains an *absolute* (or *global*) *minimum* $f(c)$ at $x = c$. An absolute maximum or an absolute minimum is an *absolute* (or *global*) *extremum*.

After all local extrema are found, we can find the absolute extrema by looking at the behaviour of the function on the ends of its domain:

Suppose function $y = f(x)$ is defined in a closed interval $[a, b]$. Find the values $f(a)$ and $f(b)$. Comparing $f(a)$ and $f(b)$ with the local maxima, if any, of $f(x)$, the greatest value is the absolute maximum of this function in $[a, b]$. Comparing $f(a)$ and $f(b)$ with the local minima, if any, of $f(x)$, and the smallest value is the absolute minimum of this function in $[a, b]$.

In this way, if a function is defined in a closed interval, there always exist an absolute maximum and an absolution minimum of this function in this interval.

If an end of an interval in the domain of this function is open, then we have to look at the behaviour of this function when the variable approaches this end. In this case absolute maximum and / or absolute minimum may not exist.

If function $y = f(x)$ that is continuous in its domain has only one local maximum, then this local maximum is the global maximum; if it has only one local minimum, then this local minimum is also the global minimum.

Example 2.1.1. (Continued) Consider function $y = x^3 + 2x^2 - 4x + 2$, $-4 \le x \le 1$.

This function attains a local maximum at $x = -2$, $y(-2) = 10$, and it attains a local minimum at $x = \frac{2}{3}$, $y\left(\frac{2}{3}\right) = \frac{14}{27} \approx 0.52$.

Since $y(-4) = -14$, $y(1) = 1$, this function attains the absolute maximum at $x = -2$, $y(-2) = 10$, and it attains the absolute minimum at $x = -4$, $y(-4) = -14$.

Example 2.1.2. Consider function $f(x) = \frac{1}{x}$, $x > 0$.

Since $\lim_{x \to 0^+} \frac{1}{x} = \infty$, $f(x)$ does not have an absolute maximum. When x approaches infinity, $\lim_{x \to \infty} \frac{1}{x} = 0$. However, $f(x)$ can be arbitrarily close to zero, but it never takes the value zero. Therefore, this function does not have an absolute minimum either.

2. SECOND DERIVATIVE ANALYSIS

Read: § 3.3.

A. Concavity of the Graph of a Function

We have seen that the sign of the second derivative determines the concavity of the graph of a function. We can use the same method as the first derivative analysis to determine the intervals where the second derivative is positives or negative:

a. Find the second derivative and find the numbers in the domain of the function where the second derivative is zero, or the second derivative does not exist. (Unfortunately, we don't have a name of such numbers. They are NOT critical numbers).

b. The numbers found in step b subdivide the domain of this function into a number of subintervals.

c. Choose a particular value c in each of the subintervals. Plug this number into the second derivative. The signs of $f''(x)$ for all x in this subinterval are the same as $f''(c)$.

d. Use the sign of the second derivative to determine the interval(s) where the graph of this function is concave up or concave down.

If the second derivative $f''(x)$ is defined in an interval around a number $x = c$, and the graph of the function $f(x)$ changes its concavity when x goes from one side of c to the other side of c, then the graph of the function has an *inflection point* at $x = c$.

Finding the concavity and inflection points of a function by its second derivative is called the *Second Derivative Analysis*.

Example 2.1.1. (Continued) Consider function $y = x^3 + 2x^2 - 4x + 2$, $-4 \le x \le 1$.

The second derivative is $y'' = 6x + 4$, which is defined everywhere in the domain of this function. Let $y'' = 0$. We have $x = -\frac{2}{3}$. When $-4 < x < -\frac{2}{3}$, $y'' < 0$, and, when $-\frac{2}{3} < x < 1$, $y'' > 0$. Hence, the graph of this function is concave down when $-4 < x < -\frac{2}{3}$, and it is concave up when $-\frac{2}{3} < x < 1$. There is an inflection point at $x = -\frac{2}{3}$, $y\left(-\frac{2}{3}\right) = \frac{142}{27} \approx 5.26$.

B. Second Derivative Test

We may also use the second derivative of a function $y = f(x)$ at a critical number $x = c$ to determine whether this function attains a local maximum or a local minimum at $x = c$.

Let $x = c$ be a critical number of function $y = f(x)$. Suppose the second derivative is defined and continuous at $x = c$. Then $f(x)$ attains a local maximum at $x = c$ if $f''(c) < 0$, and $f(x)$ attains a local minimum at $x = c$ if $f''(c) > 0$. This is called the *Second Derivative Test.*

Example 2.1.1. (Continued) Consider function $y = x^3 + 2x^2 - 4x + 2$, $-4 \le x \le 1$.

This function has critical numbers $x = -2$, and $x = \frac{2}{3}$. The second derivative is $y'' = 6x + 4$.

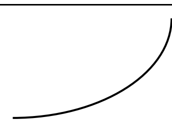
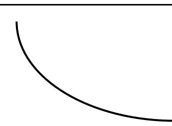
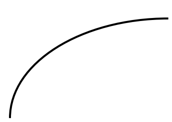
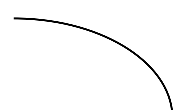
Since $y''(-2) = -8 < 0$, this function attains local maximum at $x = -2$. Since $y''\left(\frac{2}{3}\right) = 8 > 0$, this function attains a local minimum at $x = \frac{2}{3}$.

3. Graph Sketching

Read: §§ 3.4, 3.5

A. The Shape of the Graph of a Function with Known Monotonicity and Concavity

Use the first and second derivative analysis, we can find the intervals where a function $y = f(x)$ is increasing or decreasing, and intervals where the graph of the function is concave up or concave down. Putting the information together, we can find intervals where the function does not change it monotonicity and concavity. Then we know the shape of the section of the graph of this function in each of these intervals. The following is the chart again:

	increasing	decreasing
concave up		
concave down		

Then, to sketch the graph of a function, what we have to do is only to join these sections together. In this procedure, we also need the information of the asymptotes, if any, of the graph of this function.

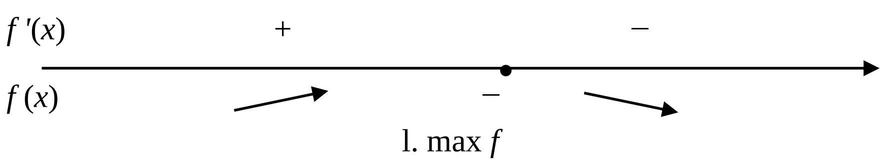
Example 2.3.1. Suppose we know the following information about a function $y = f(x)$:

(i) $f'(x) > 0$ when $x < -2$, and $f'(x) < 0$ when $x > -2$.
(ii) $f''(x) > 0$ when $x < -4$, and $f''(x) < 0$ when $x > -4$.
(iii) $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = -\infty$.
(iv) $f(0) = 0$, $f(-2) = 4$, $f(-4) = 2$.

Sketch the graph of this function.

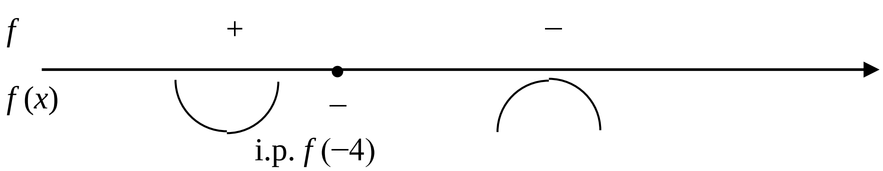
Solution. From (i), we know that this function increases when $x < -2$, it decreases when $x > -2$, and this function attains a local maximum at $x = -2$. From (iv), $f(-2) = 4$.

We have a diagram:

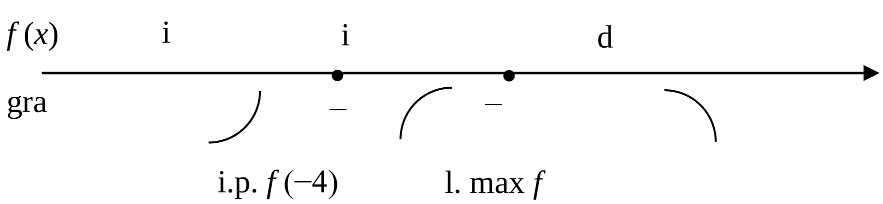


From (ii), we know the graph of this function is concave up when $x < -4$, it is concave down when $x > -4$, and it has an inflection point at $x = -4$. From (iv), $f(-4) = 2$.

Use a diagram again to illustrate:

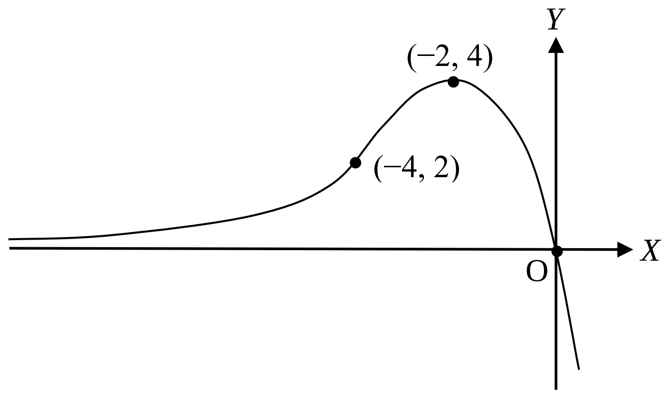


Putting the information together:



From (iii), we have a horizontal asymptote $y = 0$.

Finally, use the given value $f(0) = 0$, we have the graph of this function:



B. The Graph of a Polynomial Function

A polynomial function has the general form $y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, and $n \geq 0$ is an integer called the *degree* of the polynomial. In particular, if $n = 0$, this is a *constant function*; if $n = 1$, this is a *linear function*; if $n = 2$, this is a *quadratic function.*

The graph of a polynomial function of degree n intersects the x-axis at at most n points. If n is even, the number of intersections is even, and, if n is odd, the number of intersections is odd. For instance, the graph of a polynomial function of degree 5 may intersect the x-axis at 1, 3, or 5 points, and the graph of a polynomial function of degree 6 may intersect the x-axis at 0, 2, 4, or 6 points.

Because a polynomial is defined for all values x, the graph of a polynomial function does not have vertical asymptotes.

If $y = P(x)$ is a polynomial function of an even degree, then

$$\lim_{x\to\pm\infty} P(x) = \begin{cases} \infty & a_n > 0 \\ -\infty & a_n < 0 \end{cases}.$$

If $y = P(x)$ is a polynomial function of an odd degree, then

$$\lim_{x\to\infty} P(x) = \begin{cases} \infty & a_n > 0 \\ -\infty & a_n < 0 \end{cases}, \text{ and}$$

$$\lim_{x\to-\infty} P(x) = \begin{cases} \infty & a_n < 0 \\ -\infty & a_n > 0 \end{cases}.$$

Hence, the graph of a polynomial function does not have horizontal asymptotes.

Example 2.3.2. Consider function $y = x^3 + 2x^2 - 4x + 2$.

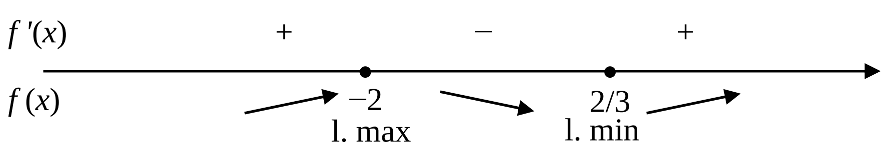
We have seen this polynomial in Example 2.1.1, but now the function is defined for all real numbers x.

Similar to Example 2.1.1, the first derivative analysis finds critical numbers $x = -2$, and $x = \frac{2}{3}$.

This function is increasing when $x < -2$, or $x > \frac{2}{3}$, and it is decreasing when $-2 < x < \frac{2}{3}$. This function attains a local maximum at $x = -2$, $y(-2) = 10$, and it attains a local minimum at $x = \frac{2}{3}$, $y\left(\frac{2}{3}\right) = \frac{14}{27} \approx 0.52$.

Since $\lim_{x\to\infty} y = \infty$, and $\lim_{x\to-\infty} y = -\infty$, this function does not have the absolute maximum or the absolute minimum.

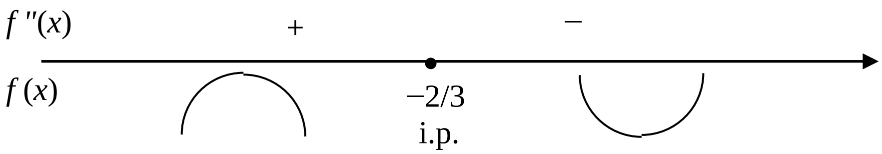
The following diagram summarizes the information:



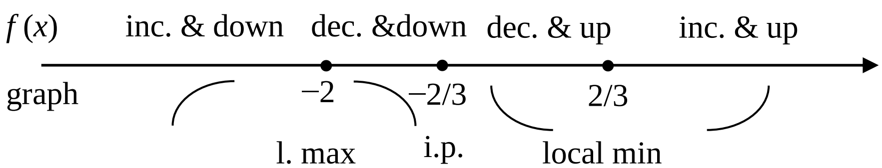
The second derivative analysis gives the following results:

The graph of this function is concave up when $x < -\frac{2}{3}$, and it is concave down when $x > -\frac{2}{3}$. This function has an inflection point at $x = -\frac{2}{3}$.

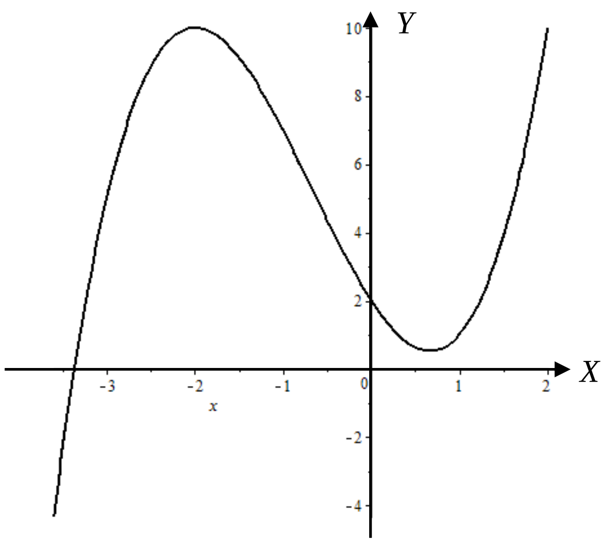
Construct another diagram as follows:



Put together:



Join the sections together. The graph of this function is



C. The Graph of a Rational Function

A rational function has the general form $y = \dfrac{P_1(x)}{P_2(x)}$, where $P_1(x)$ and $P_2(x)$ are polynomials.

The domain of this function is the set of all x such that $P_2(x) \neq 0$. We assume the fraction $\dfrac{P_1(x)}{P_2(x)}$ is irreducible. Hence, if $P_2(a) = 0$, then $P_1(a) \neq 0$. At every number a such that $P_2(a) = 0$, this function approaches infinity or negative infinity when x approaches a^+ or a^-, and the graph of this function has a vertical asymptote at $x = a$.

Let n_1 and n_2 be the degrees of $P_1(x)$ and $P_2(x)$, respectively.

If $n_1 > n_2$, then the limit of y, when x approaches infinity or negative infinity, is infinity or negative infinity. The graph of this function does not have horizontal asymptotes.

When $n_1 = n_2 = n$, let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, and $P_2(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$. then $\lim\limits_{x\to\pm\infty} \frac{P_1(x)}{P_2(x)} = \frac{a_n}{b_n}$. The graph has a horizontal asymptotes $y = \frac{a_n}{b_n}$.

When $n_1 < n_2$, $\lim\limits_{x\to\pm\infty} \frac{P_1(x)}{P_2(x)} = 0$. The graph of this function has a horizontal asymptote $y = 0$.

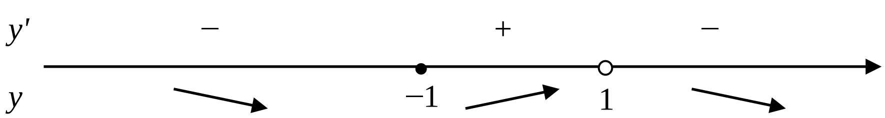
Example 2.3.3. Sketch the graph of the function $y = \frac{6x}{(x-1)^2}$.

Solution. The domain of this function consists of two open intervals $(-\infty, 1)$ and $(1, \infty)$.

First derivative analysis

The derivative of this function is $y' = -\frac{6(x+1)}{(x-1)^3}$.

The derivative does not defined at $x = 1$, but $x = 1$ is NOT a critical number because it is not in the domain of this function. Let $y' = 0$, we find a critical number $x = -1$. This critical number subdivides the domain of this function into three intervals $x < -1$, $-1 < x < 1$, and $x > 1$. Since $y'(-2) < 0$, $y'(0) > 0$, and $y'(2) < 0$, the signs of the derivative are as in the following diagram:



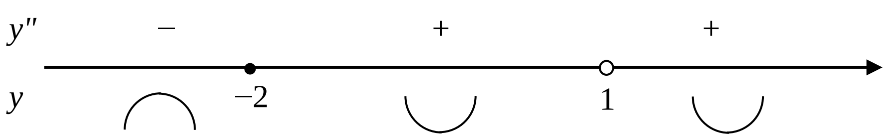
This function decreases when $x < -1$ or $x > 1$, and it increases when $-1 < x < 1$. The function has a local minimum $y(-1) = -\frac{3}{2}$.

Since $\lim_{x\to 0^+} y = \lim_{x\to 0^-} y = \infty$, this function does not have the absolute maximum. Since $x = -1$ is the only critical number when $x < 1$, $y(-1) = -\frac{3}{2}$, and $y > 0$ when $x > 1$, the local minimum of this function $y(-1) = -\frac{3}{2}$ is the absolute minimum.

Second derivative analysis

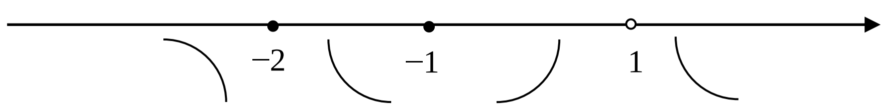
$y'' = \frac{12(x+2)}{(x-1)^4}$.

Let $y'' = 0$, $x = -2$. This number $x = -2$ subdivides the domain of this function into three subintervals $x < -2$, $-2 < x < 1$, and $x > 1$. Since $y''(-3) < 0$, $y''(0) > 0$, and $y''(2) > 0$, the signs of the second derivative are as in the following diagram:



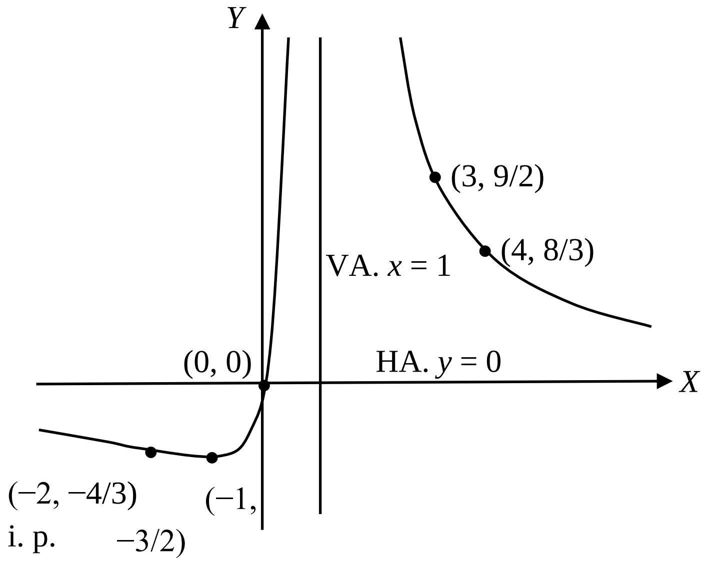
The graph of this function is concave down when $x < -2$, and it is concave up when $-2 < x < 1$ or $x > 1$. It has an inflection point $\left(-2, -\frac{4}{3}\right)$.

The shape of the graph in each interval is shown in the following diagram:



Since $\lim_{x\to 1^+} y = \lim_{x\to 1^-} y = \infty$, $x = 1$ is a vertical asymptote. Since $\lim_{x\to\infty} y = \lim_{x\to-\infty} y = 0$, $y = 0$ is a horizontal asymptote.

Find some particular points on the graph, say $y(0) = 0$, $y(3) = \frac{9}{2}$, $y(4) = \frac{8}{3}$. The graph of this function looks like the following:



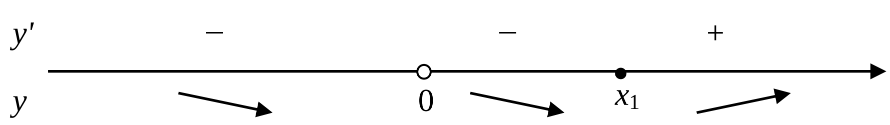
Example 2.3.4. Sketch the graph of the function $y = \dfrac{x^3+1}{x}$.

Solution. Because this is an improper rational function, i.e., the degree of the numerator is greater than the denominator, we would prefer to re-write the this function as the sum of a polynomial and a proper rational function: $y = x^2 + \dfrac{1}{x}$.

First derivative analysis

$y' = 2x - \dfrac{1}{x^2} = 0$. Let $y' = 0$. $2x^3 = 1$, $x = x_1 = 2^{-1/3} \approx 0.79$.

Since x = 0 is not in the domain of this function, although y' does not exist at $x = 0$, $x = 0$ is not a critical number. The only critical number is $x = x_1 \approx 0.79$. This critical number subdivides the domain $x \neq 0$ into three intervals $x < 0$, $0 < x < x_1$, $x > x_1$. Since $y'(-1) < 0$, $y'(0.5) < 0$, and $y'(1) > 0$, the signs of the derivative are as in the following diagram:



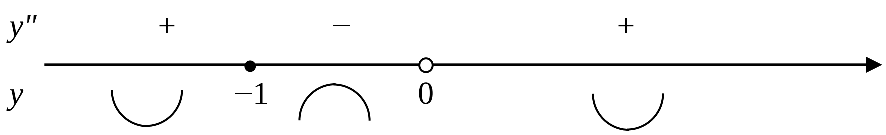
This function decreases when $x < 0$ or $0 < x < x_1$, and it increases when $x > x_1$. The function has a local minimum $y(x_1) = y_1 = 2^{-2/3} + 2^{1/3} = 3 \times 2^{-2/3} \approx 1.89$.

Since $\lim_{x\to-\infty} y = \infty$ and $\lim_{x\to 0^-} y = -\infty$, this function does not have the absolute maximum nor absolute minimum.

Second derivative analysis

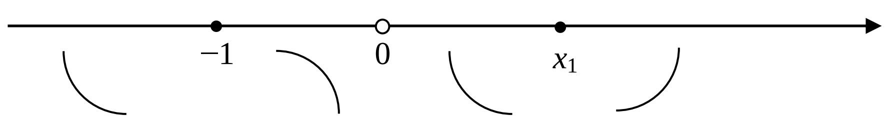
$y'' = 2 + \dfrac{2}{x^3} = 0$, Let $y'' = 0$. $x^3 = -1$, $x = -1$.

The signs of the second derivative are as in the following diagram:



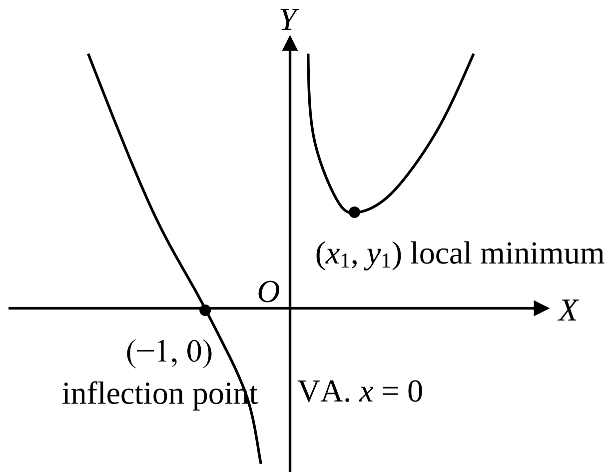
The graph of this function is concave down when $-1 < x < 0$, and it is concave up when $x < -1$ or $x > 0$. It has an inflection point $(-1, 0)$.

The shape of the graph in each interval is shown in the following diagram:



Since $\lim_{x\to 0^+} y = \infty$, $\lim_{x\to 0^-} y = -\infty$, $x = 0$ is a vertical asymptote. Since $\lim_{x\to\infty} y = \lim_{x\to-\infty} y = \infty$, this function does not have a horizontal asymptote.

The graph of this function is



4. OPTIMIZATION PROBLEMS

Read: § 3.6.

Optimization is one of the most important applications of the derivatives.

An optimization problem is to find the absolute maximum or minimum value of a quantity.

Steps of Using Calculus to Solve Applied Optimization Problems

1. Identify the quantity to be optimized. If not given, define a symbol to represent this quantity.

2. Express this quantity as a function of one or more variables. If not given, define a symbol to represent each of these variables.

3. If the number of variables is more than one, use a condition (or conditions), called the *constraint condition*(*s*), given in the question to eliminate extra variable(s) and leave the function as a one-variable function, which is called the *objective function*.

4. Find the domain of the objective function. This domain is usually NOT the natural domain of the function.

5. Use calculus to find the absolute extrema of the objective function.

Example 2.4.1. Find the maximum value of the function $z = xy^2$, $x \ge 0$, $y \ge 0$, when $x + y = 12$.

Solution. The objective function is $z = xy^2$. Since $x + y = 12$, we have $z = (12 - y)y^2 = 12y^2 - y^3$. Since $x = 12 - y \ge 0$, the domain of this function is $0 \le y \le 12$.

Let $z_y' = 24y - 3y^2 = 0$. We have $y = 0$, $y = 8$. Since $z' > 0$ when $0 < y < 8$, and $z' < 0$ when $y > 8$, function $z(y)$ attains a local maximum at $y = 8$. Then $x = 4$, and $z = 256$.

Since $z(0) = z(12) = 0$, this local maximum is also the absolute maximum.

Example 2.4.2. An open box is made by cutting four square corners of a square cardboard of dimensions 3 by 3 meters. Find the size of the square corners to be cut to maximize the volume.

Solution. Let x be the side length of the corner squares. Then the volume of the box is

$$V = (3 - 2x)^2 x = 9x - 12x^2 + 4x^3,\ 0 \le x \le \frac{3}{2}.$$

$$V' = 9 - 24x + 12x^2 = 0,\ 3 - 8x + 4x^2 = 0,\ x = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm 4}{8} = \frac{3}{2}, \frac{1}{2}.$$

By the first derivative test, V attains a local maximum at $x = \frac{1}{2}$. $V\left(\frac{1}{2}\right) = 2$. Since $V(0) = V\left(\frac{3}{2}\right) = 0$, this local maximum is also a global maximum.

Example 2.4.3. A farmer wants to use fence to enclose a rectangular region of area 800m^2 against a wall. Find the dimensions of the region that minimizes the total length of fence.

Solution. Let x be the length of the side parallel to the wall and y be the length of the side perpendicular to the wall. Then the total length of fence is $L = x + 2y$. Since the area of the region is $A = xy = 800$, $x = \frac{800}{y}$ and $L = \frac{800}{y} + 2y$. The domain of this function is $0 < y < \infty$.

Let $L' = -\frac{800}{y^2} + 2 = 0$. Then $y^2 = 400$, $y = 20$, and $x = 40$. By the first derivative test, we see that function L attains a local minimum at $x = 40$.

Since the objective function is defined in an open interval, we look at the behaviour of the function when *y* approaches 0 and *y* approaches infinity: Since $\lim_{x\to 0^+} L = \lim_{x\to\infty} L = \infty$, this local minimum is also a global minimum.

Example 2.4.4. A company wants to make an open box with a square base. The material to make the base cost 5 cents per cm^2, and the material to make the other four vertical sides cost 3 cents per cm^2. The total cost is limited to \$60. What is the largest capacity of the box?

Solution. Let the side length of the base be *x*, and the height of the box be *h*. Then the capacity $V = x^2h$. Since the total cost of the material is $C = 5x^2 + 12xh = 6000$, $xh = \frac{1}{12}(6000 - 5x^2)$.

Hence,

$$V = x(xh) = \frac{x}{12}(6000 - 5x^2) = \frac{1}{12}(6000x - 5x^3).$$

Since $h \ge 0$, we must have $6000 - 5x^2 \ge 0$. Then $x^2 \le 1200$, $x \le 20\sqrt{3}$. The domain of this function is $0 \le x \le 20\sqrt{3}$.

Let $12V' = 6000 - 15x^2 = 0$. $x^2 = 400$. $x = 20$. Then $h = \frac{50}{3}$ and $V = \frac{20000}{3}$.

Since $V' > 0$ when $x < 20$, and $V' < 0$ when $x > 20$, by the first derivative test, this function has a local maximum $V(20) = \frac{20000}{3}$. Since $V(0) = V(20\sqrt{3}) = 0$, this local maximum is also a global maximum.