

# LECTURE NOTES

MAT1339, Fall 2017

## TOPIC 1. INTRODUCING DERIVATIVES

### 1. LIMITS AND CONTINUITY

Read: §§ 1.3, 1.4.

#### A. *Limits of Sequences*

Let  $t_1, t_2, \dots, t_n, \dots$  be an infinite *sequence* of numbers, where  $t_n$  is the  $n$ -th *term* of this sequence. The expression of  $t_n$  as a function of  $n$  is the *general term* of this sequence.

If, when  $n$  becomes larger and larger,  $t_n$  goes closer and closer to a (finite) number  $L$ , then we say that, when  $n$  *approaches infinity*, the *limit* of this sequence is  $L$ , denoted by

$$\lim_{n \rightarrow \infty} t_n = L.$$

We also say that, when  $n$  approaches infinity,  $t_n$  approaches  $L$ . This sequence *converges* (or it *is convergent*) if the limit exists, otherwise, this sequence *diverges* (or it *is divergent*).

Note that, although we used the same word *approach* for both  $n$  and  $t_n$ , but " $n$  approaches infinity" means that  $n$  increases bigger and bigger without an upper bound, while " $t_n$  approaches  $L$ " means that the difference between  $t_n$  and  $L$  can be made arbitrarily small when  $n$  is large enough. Always remember that "infinity" means how a variable changes, and it is NOT a number.

*Example 1.1.1.* Consider the sequence  $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ . The general term of this sequence is

$t_n = \frac{1}{n}$ . Because, when the denominator gets larger and larger without an upper bound, the reciprocal becomes closer and closer to zero,

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

*Example 1.1.2.* Consider sequence  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots$ .

The general term of this sequence is  $t_n = \frac{n-1}{n} = 1 - \frac{1}{n}$ . When  $n$  approaches infinity,  $\frac{1}{n}$  approaches 0. Hence  $t_n$  approaches 1. We have

$$\lim_{n \rightarrow \infty} \frac{n-1}{n} = 1.$$

*Example 1.1.3.* Consider sequence  $\frac{1}{2}, -\frac{1}{2^2}, \frac{1}{2^3}, -\frac{1}{2^4}, \dots, (-1)^{n-1} \frac{1}{2^n}, \dots$ .

The general term of this sequence is  $t_n = (-1)^{n-1} \frac{1}{2^n}$ . When  $n$  approaches infinity,  $\frac{1}{2^n}$  approaches 0. Hence  $t_n$  approaches 0. We have

$$\lim_{n \rightarrow \infty} (-1)^{n-1} \frac{1}{2^n} = 0.$$

Note that, in Example 1.1.1,  $t_n$  is decreasing, in Example 1.1.2,  $t_n$  is increasing, and, in Example 1.1.3,  $t_n$  is alternatingly positive and negative.

*Example 1.1.4.* Consider the sequence 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, ... .

Since, no matter how big  $n$  is, we always have a 1 after  $t_n$ , this sequence does not have a limit.

## ***B. Limits of Functions***

Let  $y = f(x)$  be a function. If, when  $x$  becomes closer and closer to a (finite) number  $a$ , but  $x \neq a$ ,  $y$  becomes closer and closer to a (finite) number  $L$ , then we say that, when  $x$  approaches  $a$ , the *limit* of the function  $f(x)$  is  $L$ , denoted by

$$\lim_{x \rightarrow a} f(x) = L.$$

In other words, we can make the difference between  $f(x)$  and  $L$  arbitrarily small as long as the difference between  $x$  and  $a$  is small enough.

### **Limit Laws**

(i)  $\lim_{x \rightarrow a} c = c.$

(ii)  $\lim_{x \rightarrow a} x = a.$

Suppose  $\lim_{x \rightarrow a} f(x) = L_1$  and  $\lim_{x \rightarrow a} g(x) = L_2$ . Then

(iii)  $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x).$

(iv)  $\lim_{x \rightarrow a} (f(x) \pm g(x)) = L_1 \pm L_2.$

(v)  $\lim_{x \rightarrow a} (f(x)g(x)) = L_1L_2.$

(vi)  $\lim_{x \rightarrow a} (f(x)/g(x)) = L_1/L_2, L_2 \neq 0.$

(vii)  $\lim_{x \rightarrow a} (f(x))^k = L_1^k$ . If  $L_1 > 0$ , this is true for all real numbers  $k$ . If  $L_1 < 0$ , this is true for all integers  $k$ .

In particular, if  $\lim_{x \rightarrow a} f(x) \geq 0$ ,  $\lim_{x \rightarrow a} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow a} f(x)}$ .

If the limit laws cannot be used directly, we may have to make some transformations of the expression to find the limit.

### **Examples**

*Example 1.1.5.*  $\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x^3 - 8} = \lim_{x \rightarrow 2} \frac{(x+1)(x-2)}{(x-2)(x^2 + 2x + 4)} = \lim_{x \rightarrow 2} \frac{x+1}{x^2 + 2x + 4} = \frac{3}{12} = \frac{1}{4}.$

*Example 1.1.6*  $\lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{2} - \frac{1}{2+h} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{2+h-2}{2(2+h)} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{h}{2(2+h)} \right) = \lim_{h \rightarrow 0} \frac{1}{2(2+h)} = \frac{1}{4}.$

*Example 1.1.7.*

$$\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1}-2} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1}+2)}{x-3} = \lim_{x \rightarrow 3} (\sqrt{x+1}+2) = 4$$

### **One-side limits**

When  $x$  approaches  $a$  with  $x < a$ , but  $x \neq a$ ,  $f(x)$  get closer and closer to a value  $L$ , the limit is called the *left limit*, denoted by  $\lim_{x \rightarrow a^-} f(x) = L$ .

When  $x$  approaches  $a$  with  $x > a$ , but  $x \neq a$ ,  $f(x)$  get closer and closer to a value  $L$ , the limit is called the *right limit*, denoted by  $\lim_{x \rightarrow a^+} f(x) = L$ .

Limit  $\lim_{x \rightarrow a} f(x)$  exists if and only if both limits  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist and they are equal.

*Example 1.1.8.*  $u = \begin{cases} 3-x, & x > 0, \\ x-2, & x < 0. \end{cases}$  Find  $\lim_{x \rightarrow 0^-} u$  and  $\lim_{x \rightarrow 0^+} u$ .

*Solution.*  $\lim_{x \rightarrow 0^+} u = 3$ ,  $\lim_{x \rightarrow 0^-} u = -2$ .

*Example 1.1.9.*  $f(x) = \frac{x-1}{|x-1|}$ . Find  $\lim_{x \rightarrow 1^+} f(x)$  and  $\lim_{x \rightarrow 1^-} f(x)$ .

*Solution.*  $\lim_{x \rightarrow 1^+} f(x) = 1$ ,  $\lim_{x \rightarrow 1^-} f(x) = -1$ .

### **C. Limit at Infinity and Infinity as the Limit**

When  $x$  gets bigger and bigger without an upper bound, if the values of  $f(x)$  go closer and closer to a number  $L$ , then  $L$  is the limit of  $f(x)$  *at infinity*, denoted by  $\lim_{x \rightarrow \infty} f(x) = L$ .

When  $x$  gets smaller and smaller (i.e., negative with increasing absolute value) without a lower bound, if the values of  $f(x)$  go closer and closer to a number  $L$ , then  $L$  is the limit of  $f(x)$  at negative infinity, denoted by  $\lim_{x \rightarrow -\infty} f(x) = L$ .

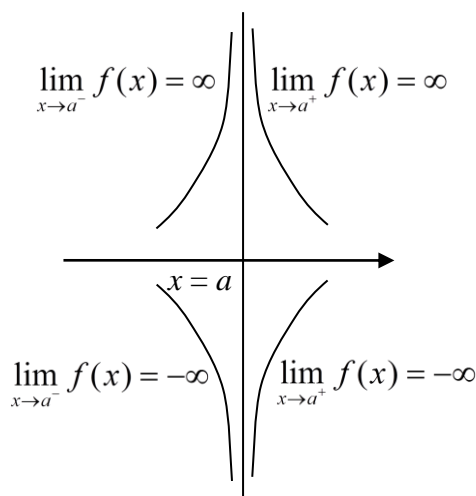
If  $\lim_{x \rightarrow \infty} f(x) = L$ , the graph of  $f(x)$  gets closer and closer to the line  $y = L$  when  $x$  approaches infinity; if  $\lim_{x \rightarrow -\infty} f(x) = L$ , the graph of  $f(x)$  gets closer and closer to the line  $y = L$  when  $x$  approaches negative infinity. In either case, the line  $y = L$  is said to be a *horizontal asymptote* of (the graph of)  $f(x)$ .

When  $x$  approaches a value  $a$  from left or right, if  $f(x)$  becomes bigger and bigger without an upper bound, then we say that the limit of  $f(x)$  is *infinity* when  $x$  approaches  $a^+$  or  $a^-$ , denoted by  $\lim_{x \rightarrow a^+} f(x) = \infty$  or  $\lim_{x \rightarrow a^-} f(x) = \infty$ .

When  $x$  approaches a value  $a$  from left or right, if  $f(x)$  becomes smaller and smaller without a lower bound, then we say that the limit of  $f(x)$  is *negative infinity* when  $x$  approaches  $a^+$  or  $a^-$ , denoted by  $\lim_{x \rightarrow a^+} f(x) = -\infty$  or  $\lim_{x \rightarrow a^-} f(x) = -\infty$ .

Note that, infinity and negative infinity are NOT numbers. Therefore, if the limit of  $f(x)$  is infinity or negative infinity when  $x$  approaches  $a^+$  or  $a^-$ , the limit DOES NOT exist.

The following figure illustrates these four cases:



In any of these cases,  $x = a$  is said to be a *vertical asymptote* of (the graph of)  $f(x)$ .

### Remarks

(a) A function may have at most two horizontal asymptotes, but it may have any number, even infinitely many, of vertical asymptotes.

(b) A horizontal asymptote may have any number, even infinitely many, intersections with the graph of the function, but a vertical asymptote will never intersect with the graph of the function.

*Example 1.1.10.* Consider function  $y = \frac{2x^2 + 1}{x^2 - x - 2}$ . Find the horizontal / vertical asymptote(s) of the graph of this function, if any.

*Solution.* Since  $\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{x^2 - x - 2} = \lim_{x \rightarrow \infty} \frac{2 + 1/x^2}{1 - 1/x - 2/x^2} = 2$ , and similarly

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 1}{x^2 - x - 2} = \lim_{x \rightarrow -\infty} \frac{2 + 1/x^2}{1 - 1/x - 2/x^2} = 2,$$

$y = 2$  is a horizontal asymptote.

Since  $x^2 - x - 2 = (x + 1)(x - 2)$ , this function is not defined when  $x = -1$  and  $x = 2$ . Consider the behaviour of this function when  $x$  approaches  $-1$  or  $2$  on each side:

Write  $y = \frac{2x^2 + 1}{(x + 1)(x - 2)}$ .

When  $x$  approaches  $-1^+$ ,  $x$  is close to  $-1$ , and it is greater than  $-1$ . Hence,  $x + 1 > 0$  and  $x - 2 < 0$ . Since the numerator is always positive,  $y$  is negative when  $x$  approaches  $-1^+$ . As the absolute value of  $x + 1$  approaches zero, we have

$$\lim_{x \rightarrow -1^+} \frac{2x^2 + 1}{x^2 - x - 2} = -\infty.$$

When  $x$  approaches  $-1^-$ ,  $x$  is close to  $-1$ , and it is less than  $-1$ . Hence,  $x + 1 < 0$  and  $x - 2 < 0$ . Since the numerator is always positive,  $y$  is positive when  $x$  approaches  $-1^-$ . As the absolute value of  $x + 1$  approaches zero, we have

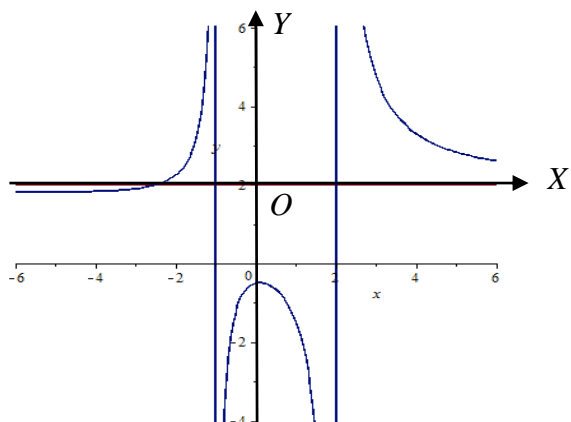
$$\lim_{x \rightarrow -1^-} \frac{2x^2 + 1}{x^2 - x - 2} = \infty.$$

Similarly, we have

$$\lim_{x \rightarrow 2^-} \frac{2x^2 + 1}{x^2 - x - 2} = -\infty, \text{ and } \lim_{x \rightarrow 2^+} \frac{2x^2 + 1}{x^2 - x - 2} = \infty.$$

Therefore,  $x = -1$  and  $x = 2$  are vertical asymptotes.

The graph of this function looks like the following:



*Example 1.1.11.* Find limit  $\lim_{x \rightarrow 2} \frac{2x^2 - x - 6}{x^2 - x - 2}$ .

*Solution.* Note that, when  $x$  approaches 2, both the top and the bottom approach 0. This means that we can factorize the top and the bottom to cancel a common factor  $x - 2$ . Indeed,  $2x^2 - x - 6 = (x - 2)(2x + 3)$ , and  $x^2 - x - 2 = (x - 2)(x + 1)$ . Hence,

$$\lim_{x \rightarrow 2} \frac{2x^2 - x - 6}{x^2 - x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(2x + 3)}{(x - 2)(x + 1)} = \lim_{x \rightarrow 2} \frac{2x + 3}{x + 1} = \frac{7}{2}.$$

Although the denominator approaches 0 when  $x$  approaches 2,  $x = 2$  is not a vertical asymptote.

### **D. Continuity**

A function  $f(x)$  is said to be *continuous* at  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ . If  $f(x)$  is continuous at every point in an interval  $I$ , then it is *continuous in  $I$* .

By definition,  $f(x)$  is not continuous at  $x = a$  if (a)  $\lim_{x \rightarrow a} f(x)$  does not exist, or (b)  $a$  is not in the domain of  $f(x)$ , or (c)  $\lim_{x \rightarrow a} f(x) \neq f(a)$ .

Intuitively,  $f(x)$  is continuous at  $x = a$  if the graph of  $f(x)$  does not "break" at this point.

### *Properties*

Suppose functions  $f(x)$  and  $g(x)$  are continuous at a point  $x = a$ . Then

- Function  $cf(x)$ , where  $c$  is a constant,  $f(x) \pm g(x)$ , and  $f(x)g(x)$ , are continuous at  $x = a$ .
- $f(x) / g(a)$  is continuous at  $x = a$  if  $g(a) \neq 0$ .
- The constant function  $y = c$ , the power function  $y = x^n$ , where  $n > 0$ , and the polynomial function  $y = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ , are continuous at all values  $x = a$ .
- The power function  $y = x^k$ , where  $k$  is a negative integer, is continuous at every  $x \neq 0$ .
- The rational functions  $y = \frac{P_1(x)}{P_2(x)}$ , where  $P_1(x)$  and  $P_2(x)$  are polynomials, are continuous at every point  $x = a$  if  $P_2(a) \neq 0$ .

*Example 1.1.12.* Consider function  $y = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$ .

Although we have  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$ , since  $x = 0$  is not in the domain of this function,  $f(x)$  is not continuous at  $x = 0$ . Function  $f(x)$  has a *hole discontinuity* at  $x = 0$ . If we add the definition  $f(0) = 0$ , this new function is continuous at  $x = 0$ . This is why a hole continuity is also called *removable discontinuity*.

*Example 1.1.13.* Consider function  $y = \begin{cases} x & x \geq 0 \\ 1 - x & x < 0 \end{cases}$ .

The value  $x = 0$  is in the domain of  $f(x)$ , but  $\lim_{x \rightarrow 0^-} f(x) = 1 \neq \lim_{x \rightarrow 0^+} f(x) = 0$ . This function is not continuous at  $x = 0$ . Function  $f(x)$  has a *jump discontinuity* at  $x = 0$ .

*Example 1.1.14.* Consider function  $y = \begin{cases} x & x \geq 0 \\ 1/x & x < 0 \end{cases}$ .

The value  $x = 0$  is in the domain of  $f(x)$ , but the  $\lim_{x \rightarrow 0^-} f(x) = -\infty$  does not exist. This function is not continuous at  $x = 0$ . Function  $f(x)$  has a *infinite discontinuity* at  $x = 0$ .

*Example 1.1.15.* Consider function  $f(x) = \begin{cases} a - x, & x < -2 \\ 2 + bx, & -2 \leq x \leq 2 \\ ax^2 + 8b, & x > 2 \end{cases}$ .

For what values of the constants  $a$  and  $b$  is  $f(x)$  continuous for all real numbers?

This function is continuous for all values  $x \neq -2$  and  $x \neq 2$ . If this function is continuous at  $x = -2$ , we must have  $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (a - x) = a + 2 = \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (2 + bx) = 2 - 2b$ , and

$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2 + bx) = 2 + 2b = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax^2 + 8b) = 4a + 8b$ . Then we have a system of two linear equations:  $a + 2 = 2 - 2b$  and  $a + 2b = 0$ . Solving this system of equations, we have  $a = 2$ ,  $b = -1$ .

## 2. INTRODUCING DERIVATIVES

Read: §§ 1.1, 1.2, 1.5, 2.3.

### A. Average Rate of Change

Let  $y = f(x)$  be a function. When  $x$  changes from  $x = a$  to  $x = b$ ,  $b - a$  is the *change* (also called the *increment*) of the variable,  $f(b) - f(a)$  is the *change* (also called the *increment*) of the function, and  $r(a, b) = \frac{f(b) - f(a)}{b - a}$  is the *average rate of change* of this function from  $x = a$  to

$x = b$ .

*Meaning:* If  $r(a, b) > 0$ , this function increases from  $x = a$  to  $x = b$ , and, if  $r(a, b) < 0$ , this function decreases from  $x = a$  to  $x = b$ . If  $|r(a, b)|$  is big, this function changes quickly from  $x = a$  to  $x = b$ , and, if  $|r(a, b)|$  is small, this function changes slowly from  $x = a$  to  $x = b$ .

#### *Interpretations*

(a) Geometric interpretation: The average rate of change of function  $y = f(x)$  from  $x = a$  to  $x = b$  is the slope of the secant line joining two points  $(a, f(a))$  and  $(b, f(b))$  on the graph of this function.

(b) Physics interpretation: If  $x = s(t)$  is the position function of the  $x$  coordinate of a particle moving along the  $x$ -axis, then the average rate of change of this function from  $t = a$  to  $t = b$  is the *average velocity* of the particle.

*Example 1.2.1.* Consider function  $f(x) = 2x - 0.1x^2$ .

$$f(2) = 3.6, f(5) = 7.5.$$

The average rate of change of this function from  $x = 2$  to  $x = 5$  is

$$r(a, b) = \frac{7.5 - 3.6}{5 - 2} = 1.3.$$

*Geometric interpretation:* The slope of the secant line joining two points  $(2, 3.6)$  and  $(5, 7.5)$  is 1.3.

*Physics interpretation:* If this function is interpreted as the position function of a particle moving along the  $x$ -axis,  $x = 2t - 0.1t^2$  (say the unit of  $x$  is meter, and the unit of  $t$  is second), then the position of this particle at time  $t = 2$  is  $x(2) = 3.6$  meters right to the origin, the position of this particle at time  $t = 5$  is  $x(5) = 7.5$  meters right to the origin, and the average velocity of the particle between  $t = 2$  and  $t = 5$  is 1.3 m/sec.

### ***B. Instantaneous Rate of Change – the Derivative***

When  $b$  approaches  $a$ , the limit of the average rate of change of function  $y = f(x)$  from  $x = a$  to  $x = b$ , if it exists, is **defined to be** the *instantaneous rate of change* of this function at  $x = a$ . This is also called the *derivative* of this function at  $x = a$ , denoted by  $f'(a)$ :

$$f'(a) = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}.$$

If write  $b = a + h$ , then  $b - a = h$ , and

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

#### *Meaning*

If  $y'(a) > 0$ , this function is increasing in an interval around  $x = a$ ; if  $y'(a) < 0$ , this function is decreasing in an interval around  $x = a$ . If  $|y'(a)|$  is small, this function changes (increases or decreases) slowly in an interval around the value  $a$ , and, if  $|y'(a)|$  is big, this function changes (increases or decreases) quickly in an interval around the value  $x = a$ .

*Interpretations*

(a) Geometric interpretation: The derivative  $f'(a)$  is the slope of the tangent line at the point  $(a, f(a))$  on the graph of this function.

Since the equation of a line with slope  $m$  going through a point  $(x_0, y_0)$  is

$$y = m(x - x_0) + y_0,$$

we can write the equation of the tangent line of the graph of function  $f(x)$  at the point  $(x_0, y_0)$  by

$$y = f'(x_0)(x - x_0) + y_0.$$

(b) Physics interpretation: If  $x = s(t)$  is the position function of the  $x$  coordinate of a particle moving along the  $x$ -axis, then the derivative of this function at  $t = a$  is the *instantaneous velocity* of the particle at  $t = a$ .

When  $a$  takes different values in the domain of a function  $y = f(x)$ , the derivative  $f'(a)$  also changes. Then we may regard the values of the derivatives with different values of  $x$  as a new function "derived" from the original function. This is the *derivative function*, or simply the *derivative* of the function, denoted by  $y'$  (to specify the variable, we may also write  $y'_x$ ), or  $f'(x)$  or  $\frac{dy}{dx}$  or  $\frac{d}{dx}f(x)$ .

The derivative of the position function of a particle moving along the  $x$ -axis is the *velocity function*.

If  $f'(x) > 0$  for all values of  $x$ ,  $a < x < b$ , then  $f(x)$  is increasing in interval  $(a, b)$ ; if  $f'(x) < 0$  for all values of  $x$ ,  $a < x < b$ , then  $f(x)$  is decreasing in interval  $(a, b)$ ;

Note that the derivative of a function at a point is a number, but the derivative of a function is another function.

We can find the derivative of some simple functions directly by the definition.

*Example 1.2.2.* Find the derivative of function  $f(x) = 2x - 0.1x^2$  at  $x = 2$  by definition.

*Solution.*  $f'(2) = 3.6$ .

$$y' = \lim_{h \rightarrow 0} \frac{(2(2+h) - 0.1(2+h)^2) - 3.6}{h} = \lim_{h \rightarrow 0} \frac{4 + 2h - 0.1(4 + 4h + h^2) - 3.6}{h} = \lim_{h \rightarrow 0} (1.6 - 0.1h) = 1.6.$$

The slope of the tangent line of the graph of  $f(x)$  at  $(2, 3.6)$  is  $m = 1.6$ , and the equation of the tangent line at this point is  $y = 1.6(x - 2) + 3.6$ , or  $y = 1.6x + 0.4$ .

If this function is the position function  $x = 2t - 0.1t^2$ , then the instantaneous velocity of this particle at  $t = 2$  is 1.6.

*Example 1.2.3.* Find the derivative of function  $y = \frac{1}{2x-3}$  by definition.

$$\begin{aligned} \text{Solution. } y' &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{2(x+h)-3} - \frac{1}{2x-3} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{(2x-3) - (2(x+h)-3)}{(2x-3)(2(x+h)-3)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-2h}{(2x-3)(2(x+h)-3)} \right) = -\frac{2}{(2x-3)^2}. \end{aligned}$$

*Example 1.2.4.* Find the derivative of the function  $y = \frac{1}{x^2-1}$  by definition. For which values of  $x$  is this function increasing, and for which values of  $x$  is this function decreasing?

*Solution.*  $y' =$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{(x+h)^2-1} - \frac{1}{x^2-1} \right) &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{(x^2-1) - ((x+h)^2-1)}{((x+h)^2-1)(x^2-1)} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{x^2-1-x^2-2xh-h^2+1}{((x+h)^2-1)(x^2-1)} \right) = \\ \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-2xh-h^2}{((x+h)^2-1)(x^2-1)} \right) &= \lim_{h \rightarrow 0} \frac{-2x-h}{((x+h)^2-1)(x^2-1)} = -\frac{2x}{(x^2-1)^2}. \end{aligned}$$

When  $x < -1$  or  $-1 < x < 0$ ,  $y' > 0$ , and, when  $0 < x < 1$  or  $x > 1$ ,  $y' < 0$ . Hence, this function increases when  $0 < x < 1$  or  $x > 1$ , and it decreases when  $x < -1$  or  $-1 < x < 0$ .

*Example 1.2.5.* Find the derivative of function  $y = \sqrt{3x-1}$  by definition.

$$\begin{aligned} \text{Solution. } y' &= \lim_{h \rightarrow 0} \frac{\sqrt{3x+3h-1} - \sqrt{3x-1}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{3x+3h-1} - \sqrt{3x-1})(\sqrt{3x+3h-1} + \sqrt{3x-1})}{h(\sqrt{3x+3h-1} + \sqrt{3x-1})} \\ &= \lim_{h \rightarrow 0} \frac{(3x+3h-1) - (3x-1)}{h(\sqrt{3x+3h-1} + \sqrt{3x-1})} = \frac{3}{2\sqrt{3x-1}}. \end{aligned}$$

Because the derivative is the limit of the average rate of change, we may use the derivative to approximate the average rate of change, or use the average rate of change to approximate the derivative.

*Example 1.2.6.* Suppose  $y = f(x)$  is a function defined in an interval around  $x = 3$ . If we know that  $f(3) = 2$  and  $f'(3) = 1.5$ , find approximation of  $f(2.8)$ .

*Solution.* Since the average rate of change  $\frac{f(2.8) - f(3)}{2.8 - 3}$  is approximately  $f'(3) = 1.5$ , we have

$$\frac{f(2.8) - f(3)}{2.8 - 3} \approx 1.5, \text{ or } f(2.8) - f(3) \approx (-0.2) \times 1.5. \quad f(2.8) \approx f(3) - 0.3 = 2 - 0.3 = 1.7.$$

*Example 1.2.7.* Suppose  $f(0) = 2$ ,  $f(2) = 3$ , and  $f(4) = 5$ . Find an approximation of  $f'(2)$ .

*Solution.* By choosing the average rate of change in different intervals, we may have three different answers to this question:

$$f'(2) \approx \frac{f(4) - f(2)}{4 - 2} = 1, \text{ or } f'(2) \approx \frac{f(2) - f(0)}{2 - 0} = \frac{1}{2}, \text{ or } f'(2) \approx \frac{f(4) - f(0)}{4 - 0} = \frac{3}{4}.$$

In practice, the last method usually gives the best result.

### C. Higher Derivatives

Since the derivative  $y' = f'(x)$  of a function  $y = f(x)$  is also a function of  $x$ , we may find the derivative of the derivative. This is the *second derivative* of  $f(x)$ , denoted by  $y''$  or  $f''(x)$ , or  $\frac{d^2 y}{dx^2}$ , or  $\frac{d^2}{dx^2} f(x)$ .





If  $f(x)$  is the position function of a particle moving along the  $x$ -axis, then  $f''(x)$  is the *acceleration* of this particle.

In a similar way, we can define other derivatives of higher orders.







If, in an interval  $(a, b)$ , the second derivative is positive, then the derivative is increasing in  $(a, b)$ . This means that the function increases faster and faster, or decreases slower and slower in  $(a, b)$ . The graph of the function looks like a part of a cup.

If, in an interval  $(a, b)$ , the second derivative is negative, then the derivative is decreasing in  $(a, b)$ . This means that the function increases slower and slower, or decreases faster and faster in  $(a, b)$ . The graph of the function looks like a part of a cap.

The following chart illustrates these situations:

	increasing	decreasing
concave up		
concave down		

The following chart is an illustration of the relation among the properties of a function and its first and second derivatives:

$f(x)$	$f'(x)$	$f''(x)$
	+	
	-	
		+
		-

### 3. DERIVATIVE FORMULAS

Read: §§ 2.1, 2.2, 2.4, 2.5.

#### A. Derivatives of Sums, differences, Products or Quotients

##### Basic Rules

$(c)' = 0$ , (the constant rule)

$(x^n)' = nx^{n-1}$ , (the power rule)

$(cf(x))' = cf'(x)$ , (the constant multiplication rule)

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$ , (the sum / difference rule)

$(f(x) g(x))' = f'(x)g(x) + f(x) g'(x)$ , (the product rule)

$(f(x)/g(x))' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ , (the quotient rule)

*Example 1.3.1.* Find the first and the second derivatives of the polynomial function  $y = 3x^3 - 2x^2 - x + 1$ .

*Solution.*  $y' = 9x^2 - 4x - 1$ ,  $y'' = 18x - 4$ .

*Example 1.3.2.* Find the derivative of  $y = \sqrt{x}(2x^2 - x)$ .

*Solution.* This question can be solved in two ways:

a. Use the product rule  $y' = (\sqrt{x})'(2x^2 - x) + \sqrt{x}(2x^2 - x)' = \frac{1}{2\sqrt{x}}(2x^2 - x) + \sqrt{x}(4x - 1)$

$$= \frac{(2x^2 - x) + 2x(4x - 1)}{2\sqrt{x}} = \frac{10x^2 - 3x}{2\sqrt{x}} = 5x\sqrt{x} - \frac{3}{2}\sqrt{x}.$$

b. Since  $y = 2x^2\sqrt{x} - x\sqrt{x} = 2x^{5/2} - x^{3/2}$ ,

$$y' = 2\left(\frac{5}{2}\right)x^{5/2-1} - \frac{3}{2}x^{3/2-1} = 5x^{3/2} - \frac{3}{2}x^{1/2} = 5x\sqrt{x} - \frac{3}{2}\sqrt{x}.$$

*Example 1.3.3.* Find the derivative of the function  $f(x) = \frac{x^3 + 3x + 1}{x^2 + 1}$ .

Use the quotient rule directly, we have

$$f'(x) = \frac{(3x^2 + 3)(x^2 + 1) - (x^3 + 3x + 1)(2x)}{(x^2 + 1)^2} = \frac{(3x^4 + 6x^2 + 3) - (2x^4 + 6x^2 + 2x)}{(x^2 + 1)^2} = \frac{x^4 - 2x + 3}{(x^2 + 1)^2}.$$

Since the degree of the top is greater than the degree of the bottom, use long division first:

$$\frac{x^3 + 3x + 1}{x^2 + 1} = x + \frac{2x + 1}{x^2 + 1}.$$

Then use the quotient rule:

$$f'(x) = 1 + \frac{2(x^2 + 1) - (2x)(2x + 1)}{(x^2 + 1)^2} = 1 + \frac{-2x^2 - 2x + 2}{(x^2 + 1)^2} = \frac{x^4 - 2x + 1}{(x^2 + 1)^2}.$$

*Example 1.3.4.* Find the derivative of the function  $y = \frac{2t^2 - 1}{\sqrt{t}}$ .

Rewrite the function with fractional exponents:  $y = 2t^{3/2} - t^{-1/2}$ . Then

$$y' = \frac{d}{dt} \left( \frac{2t^2 - 1}{\sqrt{t}} \right) = \frac{d}{dt} (2t^{3/2} - t^{-1/2}) = 2 \left( \frac{3}{2} t^{1/2} \right) + \frac{1}{2} t^{-3/2} = 3\sqrt{t} + \frac{1}{2t\sqrt{t}}.$$

We may use the quotient rule to solve this problem, but this way is easier. Always remember, if the denominator is a single power of the variable, separating the fraction into a sum of fractions is always worth trying.

### ***B. The Chain Rule***

The chain rule is used to find the derivative of a composite function. Let  $y = f(u)$  and  $u = g(x)$ . Then  $y$  is a composite function of  $x$ :  $y = f(g(x)) = (f \circ g)(x)$ . The derivative of this composite function is calculated by  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  or, simply,  $y'_x = y'_u u'_x$ .

Recall that, when we want to evaluate a function  $y = f(x)$  with a given value of  $x$ , we plug this value into this function, and, after a sequence of operations, a value of  $y$  is reached. Now look at the operation used at the last step. If the last operation is an addition, subtraction, multiplication or division, use the sum rule, difference rule, product rule or the quotient rule, respectively. Otherwise, decompose this function into a composition of two functions, and use the chain rule.

*Remark:* In the formula,  $y'_x$  is a function of  $x$ , but  $y'_u$  is a function of  $u$ . We have to use  $u = g(x)$  to write  $y'_u$  as a function of  $x$ , or, if we need the derivative at a particular value of  $x = x_0$ , plug into  $y'_u$  the value  $u_0 = g(x_0)$ .

*Examples 1.3.5.* Some values of functions  $f(x)$  and  $g(x)$ , and their derivatives  $f'(x)$  and  $g'(x)$ , are given in the following table:

$x$	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	6	-1	5	3
3	1	2	2	7
5	-3	4	3	4

If  $h(x) = (f \circ g)(x)$ , find  $h(3)$  and  $h'(3)$ ?

*Solution.* Let  $u = g(x)$ . Then  $y = h(x) = f(u)$ .  $h(3) = f(g(3)) = f(2) = 6$ .  $y_x' = y_u' u_x'$ . When  $x = 3$ ,  $u = 2$ .  $y_u' = f'(2) = -1$ ,  $u_x' = g'(3) = 7$ .  $h'(x) = y_x' = -7$ .

*Example 1.3.6.* Find the derivative of the function  $y = \sqrt{3x^2 - 2x + 1}$ .

*Solution.* Let  $u = 3x^2 - 2x - 1$ .  $u' = 6x - 2$ .  $y = \sqrt{u}$ ,  $y_u' = \frac{1}{2\sqrt{u}}$ .

$$y_x' = (6x - 2) \left( \frac{1}{2\sqrt{u}} \right) = \frac{3x - 1}{\sqrt{3x^2 - 2x + 1}}.$$

*Example 1.3.7.* Find the derivative of the function  $y = x^2 \sqrt{x^2 + 1}$ .

*Solution.* Use the product rule first, and then use the chain rule to find the derivative of  $\sqrt{x^2 + 1}$ :

$$y' = (2x)\sqrt{x^2 + 1} + x^2(\sqrt{x^2 + 1})' = 2x\sqrt{x^2 + 1} + x^2 \frac{2x}{2\sqrt{x^2 + 1}} = \frac{2x(x^2 + 1) + x^3}{\sqrt{x^2 + 1}} = \frac{3x^3 + 2x}{\sqrt{x^2 + 1}}.$$

#### 4. APPLICATIONS OF DERIVATIVES

Read: § 2.6.

##### A. Find the Equation of the Tangent Line.

The equation of a line with slope  $m$  going through a point  $(x_0, y_0)$  is  $y = m(x - x_0) + y_0$ .

At the value  $x = a$ , the slope of the tangent line of the graph of a function  $y = f(x)$  is  $f'(a)$ . Hence, the equation of the tangent line of the graph of  $f(x)$  at  $x = a$  is  $y = f'(a)(x - a) + f(a)$ .

*Example 1.4.1.* Find the equation of the tangent line of the graph of  $y = x^3 - 3x^2 + 1$  at  $x = 1$ .

*Solution.*  $y' = 3x^2 - 6x$ .  $y(1) = -1$ ,  $y'(1) = -3$ . The equation of the tangent line is  $y = -3(x - 1) - 1$ , or  $y = -3x + 2$ , or  $3x + y = 2$ .

*Example 1.4.2.* For which value(s) of  $x$  does the graph of the function  $y = x^3 - 3x^2 + 1$  have a horizontal tangent line?

*Solution.* The graph of this function has a horizontal tangent line if the derivative of this function is zero. Let  $y' = 3x^2 - 6x = 0$ . Then  $x(3x - 6) = 0$ .  $x = 0$ , or  $x = 2$ . The graph of this function has a horizontal tangent line at points  $x = 0$  and  $x = 3$ .

Let  $y = L(x)$  be the equation of the tangent line of the graph of a function  $y = f(x)$  at a values  $x = a$ . Because, near  $x = a$ , the tangent line of  $f(x)$  at this point is close to the graph of this function, we may use  $L(x)$  to be an approximation of  $f(x)$  for a value  $x$  near  $a$ .

*Example 1.4.3.* Find an approximation of  $\sqrt[3]{25}$  using the derivative of the function  $f(x) = \sqrt[3]{5x+2}$ .

*Solution.* First notice that that we have a value near 25 that has an easy-to-find cubic root, i.e.,  $\sqrt[3]{27} = 3$ . When  $5x + 2 = 27$ ,  $x = 5$ . Then we can use the equation of the tangent line of  $f(x)$  at  $x = 5$  to find an approximation of  $\sqrt[3]{25}$ . If  $5x + 2 = 25$ ,  $x = \frac{23}{5}$ . Use the chain rule,

$$f'(x) = \frac{5}{3}(5x+2)^{-2/3}. f'(5) = \frac{5}{3} \times \frac{1}{9} = \frac{5}{27}.$$

The equation of the tangent line of  $f(x)$  at  $x = 5$  is  $y = L(x)$ , where  $L(x) = \frac{5}{27}(x - 5) + 3$ . Hence,

$$\sqrt[3]{25} \approx \frac{5}{27} \left( \frac{23}{5} - 5 \right) + 3 = \frac{23}{27} - \frac{25}{27} + 3 = \frac{79}{27}.$$

## **B. Business Functions**

The *demand function* is defined by a relation between the price  $p$  of a product and the quantity  $q$  that can be sold on market with the given price. This function may be expressed by  $q$  as a function of  $p$ , or  $p$  as a function of  $q$ . This function is always decreasing.

The *supply function* is defined by a relation between the price  $p$  of a product and the quantity  $q$  that the producers would like to produce to sell at this price. This function may also be expressed by  $q$  as a function of  $p$  or  $p$  as a function of  $q$ . This function is always increasing.

The *revenue function*  $R = R(q) = D(q)q$ , where  $p = D(q)$  is the demand function, is the revenue obtained by selling  $q$  units of the product with price  $p$ .

The *cost function*  $C = C(q)$  is the cost needed to produce  $q$  units of the product.

The *profit function*  $P = P(q) = R(q) - C(q)$  is the profit obtained by selling  $q$  units of the product.

The derivative of the revenue function at  $q = a$ , called the *marginal revenue*, is (approximately) the increase of revenue by selling the next item of the product when  $q = a$ .

The derivative of the cost function at  $q = a$ , called the *marginal cost*, is (approximately) the cost needed for producing the next item of the product when  $q = a$ .

The derivative of the profit function at  $q = a$ , called the *marginal profit*, is (approximately) the profit obtained by selling the next item of the product when  $q = a$ .

*Example 1.4.4.* Suppose a supermarket can sell 150 pounds apples every day at the price \$1.2 per pound. Assume the demand function is linear, and, when the price is increased to \$1.25 per pound, the sales is dropped to 130 pound every day. If the cost of  $q$  pounds of apples is  $C(q) = 0.6q$ , find the marginal revenue function and the marginal profit function.

*Solution.* First we have to find the demand function. Since we want to express the revenue function as a function of the quantity  $q$ , we want the demand function to have the form  $p = D(q)$ . Note that, in this function,  $q$  is the variable, and  $p$  is the function. Then the slope is given by

$$m = \frac{1.25 - 1.2}{130 - 150} = -\frac{0.05}{20} = -0.0025.$$

The demand function is  $p = -0.0025(q - 150) + 1.2$ , or  $p = -0.0025q + 1.575$ .

The revenue function is  $R(q) = qD(q) = -0.0025q^2 + 1.575q$ .

The profit function is  $P(q) = R(q) - C(q) = -0.0025q^2 + 0.975q$ .

The marginal revenue function is  $R'(q) = -0.005q + 1.575$ , and the marginal profit function is  $P'(q) = -0.005q + 0.975$ .

### C. Linear Density

Suppose a rod is put along the  $x$ -axis such that one end is at the origin. The weight of a part of the rod from  $x = 0$  to a point with coordinate  $x$  is given by the function  $w(x)$ . The weight of a section of the rod from  $x = a$  to  $x = a + h$  is  $w(a + h) - w(a)$ . The *average density* of this section of the rod is

$$\bar{w}(a, a + h) = \frac{w(a + h) - w(a)}{h},$$

which is the average rate of change of the function  $w(x)$ . When  $h$  approaches zero, the derivative  $w'(a)$  is the *density* of the rod at the point  $a$ .

*Example 1.4.4.* If the weight function of a rod is given by  $w(x) = \sqrt{2x+1}$ . Find the density of the rod at  $x = 4$ .

*Solution.*  $w'(x) = \frac{1}{\sqrt{2x+1}}$ . When  $x = 4$ ,  $w'(4) = \frac{1}{3}$ .