COMP 1805 – Discrete Structures
Tutorial 11

March 28, 2013

Let A be a set and let R be a relation on A. We say that R is:

- reflexive if for all $x \in A$, $(x, x) \in R$.
- symmetric if for all $x,\ y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.
- transitive if for all $x,\ y,\ z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
- anti-symmetric if for all $x,\ y \in A$, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.
- an equivalence relation if R is reflexive, symmetric and transitive.
- a partial order if R is reflexive, anti-symmetric and transitive.
- a total order if R is a partial order and for all $x,\ y \in A$, either $(x, y) \in R$ or $(y, x) \in R$.

1. The Fibonacci numbers are defined recursively as:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Prove by induction that F_n is even if and only if n is divisible by 3.

Solution: To prove this we need to show that, for all $n \geq 0$, if n is divisible by 3 then F_n is even, and if n is not divisible by 3 then F_n is odd.

Base Case: Since the recursive definition of F_n includes two base cases, our proof must include both base cases. If $n = 0$ then $F_n = 0$. Since $n = 0$ is divisible by 3 and $F_0 = 0$ is even, the statement holds true for $n = 0$. If $n = 1$ then $F_n = 1$. Since $n = 1$ is not divisible by 3 and $F_1 = 1$ is odd, the statement holds true for $n = 1$.

Inductive Hypothesis: Assume that F_i is even if and only if i is divisible by 3 for all $0 \leq i \leq k-1$, for some integer $(k-1) \geq 1$.

Inductive Step: We need to show that F_k is even if and only if k is divisible by 3. There are two cases to consider:

- If k is divisible by 3, then $k-1$ and $k-2$ are not divisible by 3, since $k-2$, $k-1$ and k are three consecutive integers. Since $k-1$ and $k-2$ are not divisible by 3, from the inductive hypothesis we know that F_{k-1} and F_{k-2} are both odd. Then $F_k = F_{k-1} + F_{k-2}$ is even, since it is the sum of two odd integers.
- If k is not divisible by 3, then exactly one of $k-1$ and $k-2$ is divisible by 3, since $k-2$, $k-1$ and k are three consecutive integers. Then from the inductive hypothesis we know that exactly one of F_{k-1} and F_{k-2} is even (the other is odd). Then $F_k = F_{k-1} + F_{k-2}$ is odd, since it is the sum of an even and an odd integer.

Therefore, F_n is even if and only if n is divisible by 3.

2. Give an example of a relation on $A = \{a, b, c, d\}$ that is

   (a) Reflexive and symmetric.

   Solution: If the relation is reflexive, we need to have $(a, a), (b, b), (c, c), (d, d)$ in the relation. This is sufficient since this relation is also symmetric. To illustrate symmetry, though, we could also add (a, b) and (b, a) to the relation.

   (b) Symmetric but not reflexive.

   Solution: To ensure that the relation is not reflexive, we need to leave out at least one of $(a, a), (b, b), (c, c), (d, d)$. For instance, we could have $\{(a, a), (a, b), (b, a), (c, d), (d, c)\}$.

   (c) Symmetric but not transitive.

   Solution: One example is $\{(a, b), (b, a), (a, c), (c, a)\}$. This is clearly symmetric, but it is not transitive since we have (a, b) and (b, a) but not (a, a).

   (d) Transitive and symmetric but not reflexive

   Solution: One example is $\{(a, c), (c, a), (a, a), (c, c)\}$. It is clearly transitive and symmetric, but it is not reflexive because (b, b) is not in the relation.

   (e) Reflexive, antisymmetric, and transitive.

   Solution: One example is $\{(a, a), (b, b), (c, c), (d, d)\}$. It is clearly reflexive and transitive. It is antisymmetric because any time we have (x, y) and (y, x), we have $x = y$.

3. Let A be the set of all bit strings of length 4. Let R be a relation on A such that $(x, y) \in R$ if and only if the number of bits equal to 1 in x is the same as the number of bits equal to 1 in y. Prove that R is an equivalence relation.

   Solution: In order to prove that R is an equivalence relation we must show that it is reflexive, symmetric and transitive.

   - *Reflexive:* Let x be a bit string of length 4 and let n_x be the number of bits set to 1 in x. Since $n_x = n_x$ we have that $(x, x) \in R$. Therefore, R is reflexive.
   - *Symmetric:* Let x and y be bit strings of length 4 such that $(x, y) \in R$. Let n_x and n_y be the number of bits set to 1 in x and y, respectively. Since $(x, y) \in R$, we have that $n_x = n_y$, so $n_y = n_x$ and $(y, x) \in R$. Therefore, R is symmetric.
   - *Transitive:* Let x, y and z be bit strings of length 4 such that $(x, y) \in R$ and $(y, z) \in R$. Let n_x, n_y and n_z be the number of bits set to 1 in x, y and z, respectively. Since $(x, y) \in R$ and $(y, z) \in R$ we have that $n_x = n_y$ and $n_y = n_z$. Substituting n_z for n_y in the first equality, we get that $n_x = n_z$ so $(x, z) \in R$. Therefore, R is transitive.

   Therefore, since R is reflexive, symmetric and transitive, R is an equivalence relation.

4. Let $A = \{3, 5, 6, 9, 15, 24, 30, 45\}$ and let $R = \{(x, y) \mid x \text{ divides } y\}$ be a relation on A.

   (a) Show that R is a partial order.

   Solution: In order to show that R is a partial order we must show that it is reflexive, anti-symmetric and transitive.

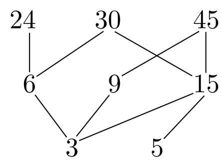
   - *Reflexive:* Since every non-zero integer divides itself we have that $(x, x) \in R$ for all $x \in A$. Therefore, R is reflexive.
   - *Anti-symmetric:* If x divides y and y divides x then there exist non-zero integers c_1 and c_2 such that $y = c_1 x$ and $x = c_2 y$. Substituting $c_1 x$ for y we get $x = c_2 c_1 x$. Since c_1 and c_2 are non-zero integers, either $c_1 = c_2 = 1$ or $c_1 = c_2 = -1$. Since A only contains positive integers, it must be the case that $c_1 = c_2 = 1$ and $x = y$. Therefore, R is anti-symmetric.

- *Transitive:* If $(x, y) \in R$ and $(y, z) \in R$ then $y = c_1 x$ for some non-zero integer c_1 and $z = c_2 y$ for some non-zero integer c_2. Substituting $c_1 x$ for y we get $z = c_2(c_1 x)$. Since $c_2 c_1$ is a non-zero integer we have that x divides z. Therefore, R is transitive.

Since R is reflexive, anti-symmetric and transitive, it is a partial order.

(b) Draw the Hasse diagram of R.

Solution:



(c) Find the maximal element(s) of R. Is there a greatest element?

Solution: An element $x \in A$ is maximal if there is no $y \in A$ such that $x \neq y$ and $(x, y) \in R$. The maximal elements of R are 24, 30 and 45. Since there are multiple maximal elements, there is no maximum element.

(d) Find the minimal element(s) of R. Is there a least element?

Solution: Similarly, an element $x \in A$ is minimal if there is no $y \in A$ such that $x \neq y$ and $(y, x) \in A$. The minimal elements of R are 3 and 5. Since there are multiple minimal elements, there is no minimum element.

(e) Find all upper bounds of $\{3, 5\}$. Is there a least upper bound?

Solution: The upper bounds of $\{3, 5\}$ are the elements $x \in A$ such that $(3, x) \in A$ and $(5, x) \in A$. The upper bounds of $\{3, 5\}$ are 15, 30 and 45. There is a least upper bound. The least upper bound is 15, since 15 divides 15, 30 and 45.

(f) Find all lower bounds of $\{15, 45\}$. Is there a greatest lower bound?

Solution: The lower bounds of $\{15, 45\}$ are the elements $x \in A$ such that $(x, 15) \in R$ and $(x, 45) \in R$. The lower bounds of $\{15, 45\}$ are 3, 5 and 15. There is a greatest lower bound. The greatest lower bound is 15, since each of 3, 5 and 15 divide 15.

(g) Find a total order on A that is compatible with R by performing a topological sort on R.

Solution: A total order R' on A is a partial order such that for all $x, y \in A$, either $(x, y) \in R'$, $(y, x) \in R'$ or both pairs are in R'. A total order R' is said to be compatible with a partial order R is $R \subseteq R'$. We can find a total order on a set A that is compatible with the partial order R by repeatedly removing a minimal element of R and placing it at the end of a list. After all of the elements are removed from R we end up with a list $a_1, \ldots, a_n$. The relation R' on A such that $(a_i, a_j) \in R'$ for all $i \leq j$ is a total order compatible with R.

In this example, one possible total order that is compatible with R comes from the list $5, 3, 15, 9, 45, 6, 24, 30$. Another possible total order that comes from the list $3, 5, 6, 9, 15, 24, 30, 45$.

5. Prove that if a relation R on a set A is symmetric then the complement $\overline{R}$ of R is also symmetric.

Solution: If R is symmetric, then $\overline{R}$ is also symmetric. We will prove this by contradiction.

Suppose $\overline{R}$ is not symmetric. Then we have some element $(a, b) \in \overline{R}$ (where $a, b \in A$) such that $(b, a) \notin \overline{R}$. By definition of the complement, $(b, a) \notin \overline{R}$ means $(b, a) \in R$. Since R is symmetric, this means that $(a, b) \in R$. But we have assumed that $(a, b) \in \overline{R}$, or $(a, b) \notin R$. So $(a, b) \in R$ and $(a, b) \notin R$, a contradiction.

6. Let A be the set of all ordered pairs of positive integers. This means that $A = \{(a, b) \mid a, b > 0\}$. Let R be a relation defined on A such that $R = \{[(a, b), (c, d)] \mid ad = bc\}$. Determine whether or not R is an equivalence relation.

   Solution: Reflexive: The relation is reflexive: $[(a, b), (a, b)] \in R$, since $ab = ba$.

   Symmetric: If $[(a, b), (c, d)] \in R$, we know that $ad = bc$. This implies that $cb = da$, i.e. $[(c, d), (a, b)] \in R$. Hence, the relation is symmetric.

   Transitive: If $[(a, b), (c, d)] \in R$ and $[(c, d), (e, f)] \in R$, we know that $ad = bc$ and $cf = de$ hold. This implies that $a = \frac{bc}{d}$ and $f = \frac{de}{c}$ and hence $af = \frac{bc}{d}\frac{de}{c} = be$. This implies that $[(a, b), (e, f)] \in R$. Hence, the relation is transitive.

   Since R is reflexive, symmetric and transitive, R is an equivalence relation.