(A)

MAT 2384 A
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
TEST #2
November 20, 2015

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:__________________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
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Signature:______________________________________

Question 1 (5 marks) Find the general solution of

$$y''' - 2y'' - 3y' = 6 + 10\cos x\,.$$

the corresponding homog. DE is $y''' - 2y'' - 3y' = 0$, which has char. eq. $\lambda^3 - 2\lambda^2 - 3\lambda = \lambda(\lambda^2 - 2\lambda - 3) = \lambda(\lambda-3)(\lambda+1) = 0$

so $\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = -1$

and then $y_h(x) = c_1 + c_2 e^{3x} + c_3 e^{-x}$

(Mod Rule needed)

$r(x) = 6 + 10\cos x \Rightarrow y_p(x) = ax + b\cos x + c\sin x$

then $y_p'(x) = a - b\sin x + c\cos x$

$y_p''(x) = -b\cos x - c\sin x,\quad y_p'''(x) = b\sin x - c\cos x$

then $y_p''' - 2y_p'' - 3y_p' = b\sin x - c\cos x - 2(-b\cos x - c\sin x)$
$- 3(a - b\sin x + c\cos x)$

$= -3a + (2b - 4c)\cos x + (4b - 2c)\sin x$

$= r(x) = 6 + 10\cos x$

so $a = -2,\quad \left.\begin{array}{l} 2b - 4c = 10 \\ 4b - 2c = 0 \end{array}\right\} \begin{array}{l} b = 1 \\ c = -2 \end{array}$

so $y_p(x) = -2x + \cos x - 2\sin x$

∴ the general solution is

$y_g(x) = y_h(x) + y_p(x)$

$= \boxed{c_1 + c_2 e^{3x} + c_3 e^{-x} - 2x + \cos x - 2\sin x}$

Question 2 (5 marks) Find the general solution of

$$y'' - 2y' + y = 2x^{3/2}\, e^x \,.$$

corresp. homog. DE is $y'' - 2y' + y = 0$, which has

char. eq. $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$

so $y_1(x) = e^x$, $y_2(x) = xe^x$ and $y_h(x) = c_1 e^x + c_2 x e^x$

$r(x) = 2x^{3/2} e^x \Rightarrow$ must use Var of Par's

$u_1' y_1 + u_2' y_2 = 0$
$u_1' y_1' + u_2' y_2' = r$
$\Rightarrow$
$u_1' e^x + u_2' x e^x = 0$
$u_1' e^x + u_2' (1+x) e^x = 2x^{3/2} e^x$

or $u_1' + u_2' x = 0$ (1)
$u_1' + u_2'(1+x) = 2x^{3/2}$ (2)

(2) − (1) $\Rightarrow u_2' = 2x^{3/2}$

then $u_2(x) = \frac{4}{5} x^{5/2}$

then (1) $\Rightarrow u_1' = -u_2' x = -2x^{5/2} \Rightarrow u_1(x) = -\frac{4}{7} x^{7/2}$

and so
$$\begin{aligned} y_p(x) &= u_1(x) y_1(x) + u_2(x) y_2(x) \\ &= \left(-\tfrac{4}{7} x^{7/2}\right)(e^x) + \left(\tfrac{4}{5} x^{5/2}\right)(xe^x) \\ &= \left(\tfrac{4}{5} - \tfrac{4}{7}\right) x^{7/2} e^x = \tfrac{8}{35} x^{7/2} e^x \end{aligned}$$

$$\therefore \boxed{y_g(x) = c_1 e^x + c_2 x e^x + \frac{8}{35} x^{7/2} e^x}$$

(A)

Question 3 (5 marks) Find the general solution of the homogeneous system

$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= y_1 + y_2 \end{aligned}.$$

we have $\vec{y}' = A\vec{y}$, where $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2 = 0$$

so $\lambda_1 = \lambda_2 = 2$

for $\lambda = 2$, we solve $(A - \lambda I)\vec{v} = \vec{0}$

or $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \vec{v} = \vec{0} \Rightarrow$ take $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then we solve $(A - \lambda I)\vec{u} = \vec{v}$

or $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ take $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore \vec{y}(x) = c_1 e^{\lambda x} \vec{v} + c_2 e^{\lambda x} (x\vec{v} + \vec{u})$$
$$= \boxed{c_1 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2x} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}$$

or
$$\boxed{\begin{aligned} y_1(x) &= c_1 e^{2x} + c_2 (x+1) e^{2x} \\ y_2(x) &= c_1 e^{2x} + c_2 x e^{2x} \end{aligned}}$$

Question 4 (5 marks) Use Gaussian Quadrature with 4 steps to approximate $\int_0^1 \cos(x^2)\,dx$ to 6 decimal places.

$$x = \frac{1}{2}\left(0(1-t) + (1)(t+1)\right) = \frac{1}{2}(t+1)$$

so $dx = \frac{1}{2}dt$

and $$\int_0^1 \cos(x^2)\,dx = \int_{-1}^{1} \cos\left(\frac{1}{4}(t+1)^2\right)\frac{1}{2}dt$$

$$\simeq \frac{1}{2}\sum_{j=1}^{4} A_j \cos\left(\frac{1}{4}(t_j+1)^2\right)$$

$$= \frac{1}{2}\Big[0.3478548451 \cos\left(\frac{1}{4}(1-0.8611363116)^2\right)$$
$$+ 0.6521451549 \cos\left(\frac{1}{4}(1-0.3399810436)^2\right)$$
$$+ 0.6521451549 \cos\left(\frac{1}{4}(1+0.3399810436)^2\right)$$
$$+ 0.3478548451 \cos\left(\frac{1}{4}(1+0.8611363116)^2\right)\Big]$$

$$= \frac{1}{2}\left(0.3478507 + 0.6482815 + 0.5875374 + 0.225379\right)$$

$$\simeq \boxed{0.904524}$$

Formulas

$$\int_a^b f(x)dx = h\sum_{j=1}^{n} f(x_j^*)\,, \qquad |\epsilon| \le \frac{1}{24}M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)|$$

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{j=1}^{n}(f(x_{j-1})+f(x_j))\,, \qquad |\epsilon| \le \frac{1}{12}M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)|$$

$$\int_a^b f(x)dx = \frac{h}{3}\sum_{j=0}^{n-1}(f(x_{2j})+4f(x_{2j+1})+f(x_{2j+2}))\,, \qquad |\epsilon| \le \frac{1}{180}M\,(b-a)\,h^4\,,$$

$$M = max_{a\le x\le b}\,|f^{(4)}(x)|$$

$$\int_{-1}^{1} f(t)\,dt \approx \sum_{j=1}^{n} A_j\,f(t_j)\,, \qquad x = \frac{1}{2}(a\,(1-t)+b\,(t+1))$$

Order n	Nodes t_j	Coefficients A_j
2	-0.5773502692	1.0
	0.5773502692	1.0
3	-0.7745966692	0.555555556
	0.0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850

(B)

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Question 1 (5 marks) Find the general solution of

$$y''' - y'' - 2y' = 10\sin x - 4\,.$$

corresp'd homog. DE $y''' - y'' - 2y' = 0$

char. eq. $\lambda^3 - \lambda^2 - 2\lambda = \lambda(\lambda^2 - \lambda - 2) = \lambda(\lambda-2)(\lambda+1) = 0$

$\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = -1$

so $y_h(x) = c_1 + c_2 e^{2x} + c_3 e^{-x}$

$r(x) = 10\sin x - 4 \Rightarrow y_p(x) = ax + b\cos x + c\sin x$

$y_p'(x) = a - b\sin x + c\cos x,\quad y_p'' = -b\cos x - c\sin x$

and $y_p''' = b\sin x - c\cos x$

$$y_p''' - y_p'' - 2y_p' = b\sin x - c\cos x - (-b\cos x - c\sin x) - 2(a - b\sin x + c\cos x)$$

$$= -2a + (b-3c)\cos x + (3b+c)\sin x$$

$$= r(x) = -4 + 10\sin x$$

$a = 2,\quad \left.\begin{array}{l} b - 3c = 0 \\ 3b + c = 10 \end{array}\right\} \quad \begin{array}{l} b = 3 \\ c = 1 \end{array}$

so $y_p(x) = 2x + 3\cos x + \sin x$

$$\therefore \boxed{y_g(x) = c_1 + c_2 e^{2x} + c_3 e^{-x} + 2x + 3\cos x + \sin x}$$

Question 2 (5 marks) Find the general solution of

$$y'' - 2y' + y = 3x^{5/2}\, e^x\,.$$

$y_h(x) = c_1 e^x + c_2 x e^x$

$u_1' y_1 + u_2' y_2 = 0$ $\Rightarrow$ $u_1' e^x + u_2' x e^x = 0$

$u_1' y_1' + u_2' y_2' = r$ $\quad$ $u_1' e^x + u_2'(x+1) e^x = 3x^{5/2} e^x$

or $u_1' + u_2' x = 0$ ① $\quad$ ② − ① $\Rightarrow$ $u_2' = 3x^{5/2}$

$u_1' + u_2'(x+1) = 3x^{5/2}$ ②

so $u_2(x) = \frac{6}{7} x^{7/2}$

then ① $\Rightarrow$ $u_1' = -u_2' x = -3x^{7/2} \Rightarrow u_1(x) = -\frac{2}{3} x^{9/2}$

then $y_p(x) = \left(-\frac{2}{3} x^{9/2}\right)(e^x) + \left(\frac{6}{7} x^{7/2}\right)(x e^x)$

$= \left(-\frac{2}{3} + \frac{6}{7}\right) x^{9/2} e^x$

$= \frac{4}{21} x^{9/2} e^x$

$\therefore$ $\boxed{y_g(x) = c_1 e^x + c_2 x e^x + \frac{4}{21} x^{9/2} e^x}$

Question 3 (5 marks) Find the general solution of the homogeneous system

$$\begin{aligned} y_1' &= -y_1 - y_2 \\ y_2' &= y_1 - 3y_2 \end{aligned}.$$

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda)+1$$
$$= \lambda^2 + 4\lambda + 4$$
$$= (\lambda+2)^2 = 0$$

so $\lambda_1 = \lambda_2 = -2$

$(A - \lambda I)\vec{v} = \vec{0}$ is $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow$ take $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A - \lambda I)\vec{u} = \vec{v}$ is $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ take $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $\boxed{\vec{y}(x) = C_1 e^{-2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2x}\left(x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$

or
$$\boxed{\begin{aligned} y_1(x) &= C_1 e^{-2x} + C_2(x+1)e^{-2x} \\ y_2(x) &= C_1 e^{-2x} + C_2 x e^{-2x} \end{aligned}}$$

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$$x = \frac{1}{2}(t+1), \qquad dx = \frac{1}{2}dt$$

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$$\simeq \frac{1}{2}\sum_{j=1}^4 A_j \sin\left(\frac{1}{4}(t_j+1)^2\right)$$

$$= \frac{1}{2}\Big[0.347854845\,1 \sin\left(\frac{1}{4}(1-0.8611363116)^2\right)$$

$$+ 0.6521451549 \sin\left(\frac{1}{4}(1-0.3399810436)^2\right)$$

$$+ 0.6521451549 \sin\left(\frac{1}{4}(1+0.3399810436)^2\right)$$

$$+ 0.3478548451 \sin\left(\frac{1}{4}(1+0.8611363116)^2\right)\Big]$$

$$= \frac{1}{2}\left[0.0016769 + 0.0679477 + 0.283007 + 0.2649664\right]$$

$$\simeq \boxed{0.308799}$$

Formulas

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$$\int_a^b f(x)dx = \frac{h}{3}\sum_{j=0}^{n-1}(f(x_{2j})+4f(x_{2j+1})+f(x_{2j+2}))\,, \qquad |\epsilon| \le \frac{1}{180} M\,(b-a)\,h^4\,,$$
$$M = max_{a\le x\le b}\,|f^{(4)}(x)|$$

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