MAT 2384 3X
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
TEST #1
May 31, 2017

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:________________

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 5 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
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Signature:________________

Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $\cos y\, y' = \dfrac{2e^x}{1+e^x}$ DE is separable

$$\int \cos y \, dy = \int \frac{2e^x}{1+e^x}\, dx + C$$

$$\sin y = 2\ln(1+e^x) + C$$

so $\boxed{y = \arcsin(2\ln(1+e^x) + C)}$

(b) $y' + \dfrac{4}{x}y = 4x^2$ DE is linear with $f(x) = \frac{4}{x}$, $r(x) = 4x^2$

so $\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$

and then $y = \dfrac{1}{x^4}\left[\int (x^4)(4x^2)\,dx + C\right]$

$$= x^{-4}\left(\int 4x^6 dx + C\right)$$

$$= x^{-4}\left(\frac{4}{7}x^7 + C\right)$$

$$= \boxed{\frac{4}{7}x^3 + Cx^{-4}}$$

Question 2 (5 marks) Find the general solution:

$$(12xy^2 - 5y^2\sin x)\,dx + (18x^2y + 15y\cos x + y^2)\,dy = 0$$ (not separable)

$M(x,y) = 12xy^2 - 5y^2\sin x \Rightarrow M_y = 24xy - 10y\sin x$

$N(x,y) = 18x^2y + 15y\cos x + y^2 \Rightarrow N_x = 36xy - 15y\sin x$

$\Big\}$ $M_y \neq N_x$ DE not exact

$\dfrac{M_y - N_x}{M} = \dfrac{-12xy + 5y\sin x}{12xy^2 - 5y^2\sin x} = \dfrac{-1}{y}$ a function of y only

then $\mu(y) = e^{-\int -\frac{1}{y}dy} = e^{\ln y} = y$ and the DE becomes

$$(12xy^3 - 5y^3\sin x)\,dx + (18x^2y^2 + 15y^2\cos x + y^3)\,dy = 0$$

$M^*(x,y) = 12xy^3 - 5y^3\sin x \Rightarrow M^*_y = 36xy^2 - 15y^2\sin x$

$N^*(x,y) = 18x^2y^2 + 15y^2\cos x + y^3 \Rightarrow N^*_x = 36xy^2 - 15y^2\sin x$

$\Big\}$ $M^*_y = N^*_x$ DE now exact

then $F(x,y) = \int N^*(x,y)\,dy + g(x) = \int (18x^2y^2 + 15y^2\cos x + y^3)\,dy + g(x)$

$= 6x^2y^3 + 5y^3\cos x + \frac{1}{4}y^4 + g(x)$

so $\dfrac{\partial F}{\partial x} = 12xy^3 - 5y^3\sin x + g'(x) = M^*(x,y) = 12xy^3 - 5y^3\sin x$

so $g'(x) = 0 \Rightarrow$ take $g(x) = 0$

thus $F(x,y) = 6x^2y^3 + 5y^3\cos x + \frac{1}{4}y^4$

and the general solution is $\boxed{6x^2y^3 + 5y^3\cos x + \frac{1}{4}y^4 = C}$

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' - 2\sqrt{3}y' + 3y = 0$, $y(0) = 0$, $y'(0) = \sqrt{3}$

the characteristic equation is $\lambda^2 - 2\sqrt{3}\lambda + 3 = (\lambda - \sqrt{3})^2 = 0$

so $\lambda_1 = \lambda_2 = \sqrt{3}$ and the general solution is

$y(x) = c_1 e^{\sqrt{3}x} + c_2 x e^{\sqrt{3}x}$

$y(0) = 0 \Rightarrow c_1 e^0 + c_2 (0) e^0 = 0 \Rightarrow c_1 = 0$

$y'(x) = \sqrt{3} c_1 e^{\sqrt{3}x} + c_2 e^{\sqrt{3}x} + \sqrt{3} c_2 x e^{\sqrt{3}x}$

$y'(0) = \sqrt{3} \Rightarrow \sqrt{3} c_1 e^0 + c_2 e^0 + \sqrt{3} c_2 (0) e^0 = \sqrt{3} \Rightarrow c_2 = \sqrt{3}$

$\therefore$ the unique solution is $\boxed{y(x) = \sqrt{3}\, x e^{\sqrt{3}x}}$

(b) $y'' - 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = 4$

the char. eq. is $\lambda^2 - 2\lambda + 5 = 0$

So $\lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

the general solution is $y(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$

$y(0) = 2 \Rightarrow c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = 2 \Rightarrow c_1 = 2$

$y'(x) = c_1 e^x \cos(2x) - 2c_1 e^x \sin(2x) + c_2 e^x \sin(2x) + 2c_2 e^x \cos(2x)$

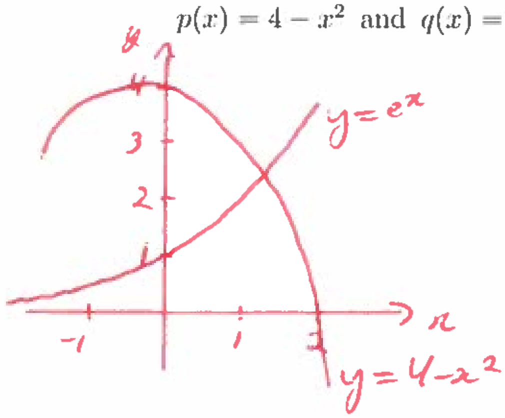
$y'(0) = 4 \Rightarrow c_1 e^0 \cos(0) - 2c_1 e^0 \sin(0) + c_2 e^0 \sin(0) + 2c_2 e^0 \cos(0) = 4$

or $c_1 + 2c_2 = 4 \Rightarrow c_2 = 1$

$\therefore$ the unique solution is

$$\boxed{y(x) = 2e^x \cos(2x) + e^x \sin(2x)}$$

Question 4 (5 marks) Use Newton's Method to find the positive point of intersection of $p(x) = 4 - x^2$ and $q(x) = e^x$ to 6 decimal places. Start with $x_0 = 1$.



Let $f(x) = 4 - x^2 - e^x$

then $f'(x) = -2x - e^x$

Newton's Method
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4 - x_n^2 - e^{x_n}}{-2x_n - e^{x_n}} = x_n - \frac{x_n^2 + e^{x_n} - 4}{2x_n + e^{x_n}}$$

$x_0 = 1$ $\quad x_1 = 1 - \frac{e - 3}{2 + e} = 1.059708$

$x_2 = 1.058008$

$x_3 = 1.058006 = x_4 \quad \therefore$ stop

(check: $p(1.058006) \simeq 2.880623$
$q(1.058006) \simeq 2.880621$ okay! :))

$\therefore$ functions intersect at $\boxed{x = 1.058006}$

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Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $\cos y\, y' = \dfrac{3e^x}{2+e^x}$

DE is separable

$$\int \cos y\, dy = \int \frac{3e^x}{2+e^x}\, dx + C$$

$$\sin y = 3\ln(2+e^x) + C$$

$$\boxed{y = \arcsin(3\ln(2+e^x) + C)}$$

(b) $y' + \dfrac{3}{x}y = 3x^3$

DE is linear with $f(x) = \frac{3}{x}$, $r(x) = 3x^3$

so $\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

then $y = \frac{1}{x^3}\left[\int (x^3)(3x^3)\, dx + C\right]$

$= x^{-3}\left(\int 3x^6 + C\right)$

$= x^{-3}\left(\frac{3}{7}x^7 + C\right)$

$= \boxed{\frac{3}{7}x^4 + Cx^{-3}}$

Question 2 (5 marks) Find the general solution:

$$(15x^2y^3 + 2y^3\cos x)\,dx + (20x^3y^2 + 8y^2\sin x + 3y)\,dy = 0$$ (not separable)

$M(x,y) = 15x^2y^3 + 2y^3\cos x \Rightarrow M_y = 45x^2y^2 + 6y^2\cos x$
$N(x,y) = 20x^3y^2 + 8y^2\sin x + 3y \Rightarrow N_x = 60x^2y^2 + 8y^2\cos x$
$\Big\}$ $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{M} = \frac{-15x^2y^2 - 2y^2\cos x}{15x^2y^3 + 2y^3\cos x} = \frac{-1}{y}$$ function of y only

so $\mu(y) = e^{-\int \frac{-1}{y}dy} = e^{\ln y} = y$ and the DE becomes

$$(15x^2y^4 + 2y^4\cos x)\,dx + (20x^3y^3 + 8y^3\sin x + 3y^2)\,dy = 0$$

$M^*(x,y) = 15x^2y^4 + 2y^4\cos x \Rightarrow M^*_y = 60x^2y^3 + 8y^3\cos x$
$N^*(x,y) = 20x^3y^3 + 8y^3\sin x + 3y^2 \Rightarrow N^*_x = 60x^2y^3 + 8y^3\cos x$
$\Big\}$ $M^*_y = N^*_x$ DE exact

$$F(x,y) = \int N^*(x,y)\,dy + g(x) = \int (20x^3y^3 + 8y^3\sin x + 3y^2)\,dy + g(x)$$
$$= 5x^3y^4 + 2y^4\sin x + y^3 + g(x)$$

then $\frac{\partial F}{\partial x} = 15x^2y^4 + 2y^4\cos x + g'(x) = M^*(x,y) = 15x^2y^4 + 2y^4\cos x$

so $g'(x) = 0 \Rightarrow$ take $g(x) = 0$

thus $F(x,y) = 5x^3y^4 + 2y^4\sin x + y^3$

$\therefore$ the general solution is

$$\boxed{5x^3y^4 + 2y^4\sin x + y^3 = C}$$

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' - 2\sqrt{5}\,y' + 5y = 0$, $y(0) = 0$, $y'(0) = \sqrt{5}$

the char. eq. is $\lambda^2 - 2\sqrt{5}\lambda + 5 = (\lambda - \sqrt{5})^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \sqrt{5}$

the general solution is $y(x) = C_1 e^{\sqrt{5}x} + C_2 x e^{\sqrt{5}x}$

$y(0) = 0 \Rightarrow C_1 e^0 + C_2 (0) e^0 = 0 \Rightarrow C_1 = 0$

$y'(x) = \sqrt{5} C_1 e^{\sqrt{5}x} + C_2 e^{\sqrt{5}x} + \sqrt{5} C_2 x e^{\sqrt{5}x}$

$y'(0) = \sqrt{5} \Rightarrow \sqrt{5} C_1 e^0 + C_2 e^0 + \sqrt{5} C_2 (0) e^0 = \sqrt{5} \Rightarrow C_2 = \sqrt{5}$

$\therefore$ the unique solution is $\boxed{y(x) = \sqrt{5}\, x\, e^{\sqrt{5}x}}$

(b) $y'' - 2y' + 10y = 0$, $y(0) = 3$, $y'(0) = 9$

the char. eq. is $\lambda^2 - 2\lambda + 10 = 0$

so $\lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2} = \dfrac{2 \pm \sqrt{-36}}{2} = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$

So the general solution is $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$

$y(0) = 3 \Rightarrow C_1 e^0 \cos(0) + C_2 e^0 \sin(0) = 3 \Rightarrow C_1 = 3$

$y'(x) = C_1 e^x \cos(3x) - 3C_1 e^x \sin(3x) + C_2 e^x \sin(3x) + 3C_2 e^x \cos(3x)$

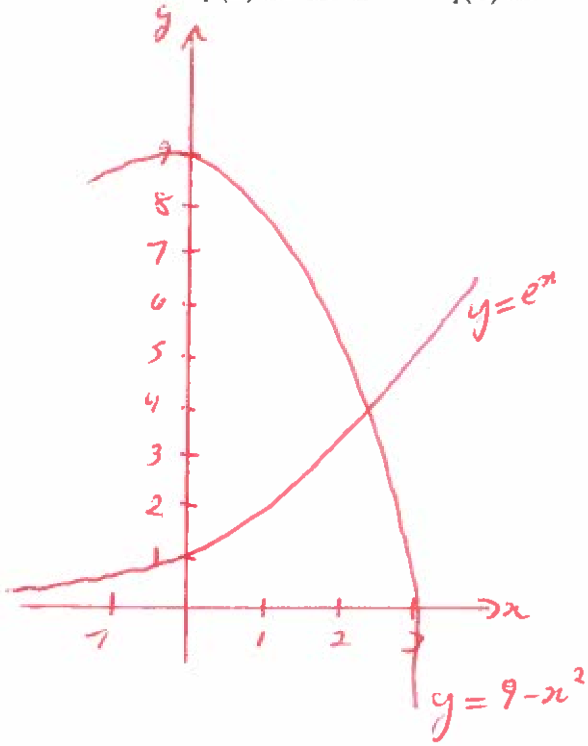
$y'(0) = 9 \Rightarrow C_1 e^0 \cos(0) - 3C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 3C_2 e^0 \cos(0) = 9$

or $C_1 + 3C_2 = 9 \Rightarrow C_2 = 2$

$\therefore$ the unique solution is

$\boxed{y(x) = 3e^x \cos(3x) + 2e^x \sin(3x)}$

Question 4 (5 marks) Use Newton's Method to find the positive point of intersection of $p(x) = 9 - x^2$ and $q(x) = e^x$ to 6 decimal places. Start with $x_0 = 2$.



Let $f(x) = e^x - 9 + x^2$

then $f'(x) = e^x + 2x$

$$x_{n+1} = x_n - \frac{e^{x_n} + x_n^2 - 9}{e^{x_n} + 2x_n}$$

$$x_0 = 2 \Rightarrow x_1 = 2 - \frac{e^2 - 5}{e^2 + 4} = 1.790232$$

$$x_2 = 1.769778$$

$$x_3 = 1.769601 = x_4 \quad \therefore \text{stop}$$

(Check: $p(1.769601) \simeq 5.868512$
$q(1.769601) \simeq 5.868511$ okay! :))

$\therefore$ the point of intersection is at $\boxed{x = 1.769601}$