MAT 2384 C Assignment #7 Solutions

1. $\mathcal{L}\{e^{-5t}\cos(2t)\} = F(s+5) = \boxed{\dfrac{s+5}{(s+5)^2+4}} = \boxed{\dfrac{s+5}{s^2+10s+29}}$

$\left(F(s) = \mathcal{L}\{\cos(2t)\} = \dfrac{s}{s^2+4} \right)$

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2. $\mathcal{L}\{4e^{2t}t^3\} = F(s-2) = \boxed{\dfrac{24}{(s-2)^4}}$

$\left(F(s) = \mathcal{L}\{4t^3\} = \dfrac{24}{s^4} \right)$

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3. $\mathcal{L}\{u(t-3)(3t^2-5t+4)\} = e^{-3s}\mathcal{L}\{3t^2+13t+16\}$

$$\left(\begin{aligned} f(t-3) &= 3t^2-5t+4 \\ f(t) &= 3(t+3)^2-5(t+3)+4 \\ &= 3(t^2+6t+9)-5(t+3)+4 \end{aligned} \right) = \boxed{e^{-3s}\left(\frac{6}{s^3}+\frac{13}{s^2}+\frac{16}{s}\right)}$$

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4. $\mathcal{L}\{u(t-\pi)(4\cos(3t)-2\sin t)\} = e^{-\pi s}\mathcal{L}\{2\sin t - 4\cos(3t)\}$

$$\left(\begin{aligned} f(t-\pi) &= 4\cos(3t)-2\sin t \\ f(t) &= 4\cos(3(t+\pi))-2\sin(t+\pi) \\ &= 4\cos(3t+3\pi)-2\sin(t+\pi) \\ &= -4\cos(3t)+2\sin t \end{aligned} \right) = \boxed{e^{-\pi s}\left(\frac{2}{s^2+4}-\frac{4s}{s^2+9}\right)}$$

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5. $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{5}{(s+1)^2+4}\right\} = e^{-t}f(t)$

$\left(F(s) = \dfrac{5}{s^2+4} \Rightarrow f(t) = \dfrac{5}{2}\sin(2t) \right) = \boxed{\dfrac{5}{2}e^{-t}\sin(2t)}$

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6. $\mathcal{L}^{-1}\left\{e^{-4s}\left(\dfrac{6}{s^3}+\dfrac{3}{s^2}\right)\right\} = u(t-4)f(t-4)$

$= u(t-4)\left(3(t-4)^2+3(t-4)\right)$

$\left(f(t) = \mathcal{L}^{-1}\left\{\dfrac{6}{s^3}+\dfrac{3}{s^2}\right\} = 3t^2+3t \right) = \boxed{u(t-4)(3t^2-21t+36)}$

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7. $$y'' - y' - 6y = \begin{cases} 0 & 0<t<2 \\ e^t & t>2 \end{cases} = u(t-2)e^t, \quad \begin{matrix} y(0)=3 \\ y'(0)=4 \end{matrix}$$

let $Y(s) = \mathcal{L}\{y(t)\}$ and take the Transform of the DE

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{u(t-2)e^t\} \qquad (f(t-2)=e^t)$$

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 6Y(s) = e^{-2s}\mathcal{L}\{e^{t+2}\}$$

$$(s^2 - s - 6)Y(s) - 3s - 4 + 3 = e^2 e^{-2s}\frac{1}{s-1}$$

$$(s^2 - s - 6)Y(s) = 3s + 1 + e^2 e^{-2s}\frac{1}{s-1}$$

$$Y(s) = \frac{3s+1}{(s-3)(s+2)} + e^2 e^{-2s}\frac{1}{(s-1)(s-3)(s+2)}$$

$$\left(\frac{3s+1}{(s-3)(s+2)} = \frac{a}{s-3} + \frac{b}{s+2} \qquad \left.\begin{matrix} a+b=3 \\ 2a-3b=1 \end{matrix}\right\} \quad \begin{matrix} a=2 \\ b=1 \end{matrix} \right)$$

$$\left(\frac{1}{(s-1)(s-3)(s+2)} = \frac{c}{s-1} + \frac{d}{s-3} + \frac{f}{s+2} \qquad \left.\begin{matrix} c+d+f=0 \\ -c+d-4f=0 \\ -6c-2d+3f=1 \end{matrix}\right\} \quad \left.\begin{matrix} 2d-3f=0 \\ 4d+9f=1 \end{matrix}\right\} \begin{matrix} d=1/10 \\ f=1/15 \end{matrix} \quad \text{then } c=-1/6 \right)$$

so $$Y(s) = \frac{2}{s-3} + \frac{1}{s+2} + e^2 e^{-2s}\left(-\frac{1}{6}\left(\frac{1}{s-1}\right) + \frac{1}{10}\left(\frac{1}{s-3}\right) + \frac{1}{15}\left(\frac{1}{s+2}\right) \right)$$

∴ the solution of the IVP is

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2e^{3t} + e^{-2t} + e^2 u(t-2)\left(-\frac{1}{6}e^{t-2} + \frac{1}{10}e^{3(t-2)} + \frac{1}{15}e^{-2(t-2)} \right)}$$

8. $y'' + 4y = \delta(t-\pi)$, $y(0) = -2$, $y'(0) = 8$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and take the Transform of the DE

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$

$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = e^{-\pi s}$

$(s^2+4)Y(s) = -2s + 8 + e^{-\pi s}$

$$Y(s) = \frac{-2s+8}{s^2+4} + \frac{e^{-\pi s}}{s^2+4}$$

$\therefore$ the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4\sin(2t) - 2\cos(2t) + \frac{1}{2}u(t-\pi)\sin(2(t-\pi))$$

$$= \boxed{4\sin(2t) - 2\cos(2t) + \frac{1}{2}u(t-\pi)\sin(2t)}$$

or

$$\boxed{y(t) = \begin{cases} 4\sin(2t) - 2\cos(2t) & 0 < t < \pi \\ 9/2\,\sin(2t) - 2\cos(2t) & t > \pi \end{cases}}$$

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9. $y' = 4x + 2y$; $y(0) = 0$

(the true solution is $y(x) = e^{2x} - 2x - 1$)

$h = 0.2 \Rightarrow n = 5$ steps $\Rightarrow x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, etc...

Improved Euler's Method

$$y_{n+1}^{p} = y_n^c + h f(x_n, y_n^c) = y_n^c + (0.2)(4x_n + 2y_n^c)$$

$$= 0.8x_n + 1.4y_n^c$$

$$y_{n+1}^{c} = y_n^c + \frac{1}{2}h\left(f(x_n, y_n^c) + f(x_{n+1}, y_{n+1}^{p})\right)$$

$$= y_n^c + (0.1)(4x_n + 2y_n^c + 4x_{n+1} + 2y_{n+1}^{p})$$

$$= y_n^c + 0.4x_n + 0.2y_n^c + 0.4x_{n+1} + (0.2)(0.8x_n + 1.4y_n^c)$$

$$= 1.48\, y_n^c + 0.56\, x_n + 0.4\, x_{n+1}$$

then
$$y_1^c = 1.48\, y_0 + 0.56\, x_0 + 0.4\, x_1$$
$$= 1.48(0) + 0.56(0) + (0.4)(0.2) = 0.08$$

$$y_2^c = 1.48\, y_1^c + 0.56\, x_1 + 0.4\, x_2$$
$$= 1.48(0.08) + 0.56(0.2) + 0.4(0.4) = 0.3904$$

$$y_3^c = 1.48\, y_2^c + 0.56\, x_2 + 0.4\, x_3$$
$$= 1.48(0.3904) + 0.56(0.4) + 0.4(0.6) = 1.0418$$

$$y_4^c = 1.48\, y_3^c + 0.56\, x_3 + 0.4\, x_4$$
$$= 1.48(1.0418) + 0.56(0.6) + 0.4(0.8) = 2.1979$$

$$y_5^c = 1.48\, y_4^c + 0.56\, x_4 + 0.4\, x_5$$
$$= 1.48(2.1979) + 0.56(0.8) + 0.4(1) = 4.1009$$

n	x_n	y_n^c	$y(x_n)$	error	(%) relative error
1	0.2	0.0800	0.0918	0.0118	12.9
2	0.4	0.3904	0.4255	0.0351	8.2
3	0.6	1.0418	1.1201	0.0783	7.0
4	0.8	2.1979	2.3530	0.1551	6.6
5	1.0	4.1009	4.3891	0.2882	6.6

(results not great, but better than Euler)