MAT 2384 A Assignment # 6 Solutions

1. $y_1' = 4y_1 - 2y_2 - 2x - 5, \quad y_1(0) = 2$
$y_2' = 3y_1 - y_2 - 2x - 3, \quad y_2(0) = 2$

so we have $\vec{y}\,' = A\vec{y} + \vec{F}(x)$, where $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$, $\vec{F}(x) = \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix}$ and $\vec{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

the corresponding homogeneous system is $\vec{y}\,' = A\vec{y}$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$

for $\lambda_1 = 1$, $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$ is $\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

for $\lambda_2 = 2$, $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$ is $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and so $\vec{y}_h(x) = c_1 e^{x} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{F}(x) = \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix} \Rightarrow$ take $\vec{y}_p(x) = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} \Rightarrow \vec{y}_p\,'(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

and $A\vec{y}_p + \vec{F} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} + \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix}$

$$= \begin{bmatrix} (4a_1 - 2a_2 - 2)x + 4b_1 - 2b_2 - 5 \\ (3a_1 - a_2 - 2)x + 3b_1 - b_2 - 3 \end{bmatrix}$$

$$= \vec{y}_p\,' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

means that $\left.\begin{array}{l} 4a_1 - 2a_2 - 2 = 0 \\ 3a_1 - a_2 - 2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4a_1 - 2a_2 = 2 \\ 3a_1 - a_2 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = 1 \\ a_2 = 1 \end{array}$

and $\left.\begin{array}{l} 4b_1 - 2b_2 - 5 = a_1 = 1 \\ 3b_1 - b_2 - 3 = a_2 = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4b_1 - 2b_2 = 6 \\ 3b_1 - b_2 = 4 \end{array}\right\}$ $\begin{array}{l} b_1 = 1 \\ b_2 = -1 \end{array}$

and the particular solution is $\bar{y}_p(x) = \begin{bmatrix} x+1 \\ x-1 \end{bmatrix}$

so the general solution is

$$\bar{y}_g(x) = c_1 e^x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x+1 \\ x-1 \end{bmatrix}$$

$$\bar{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow c_1 e^0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

or $\left.\begin{array}{l} 2c_1 + c_2 = 1 \\ 3c_1 + c_2 = 3 \end{array}\right\}$ $c_1 = 2 \Rightarrow c_2 = -3$

$\therefore$ the unique solution is

$$\boxed{\begin{aligned} \bar{y}(x) &= 2e^x \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x+1 \\ x-1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^x - 3e^{2x} + x + 1 \\ 6e^x - 3e^{2x} + x - 1 \end{bmatrix} \end{aligned}}$$

or $\boxed{y_1(x) = 4e^x - 3e^{2x} + x + 1 \quad \text{and} \quad y_2(x) = 6e^x - 3e^{2x} + x - 1}$

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2. $\begin{array}{ll} y_1' = 2y_1 - 4y_2 + 10x - 2x^2, & y_1(0) = 2 \\ y_2' = 4y_1 - 6y_2 + 2 + 12x - 4x^2, & y_2(0) = 1 \end{array}$

$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$, $\vec{F}(x) = \begin{bmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{bmatrix}$, $\bar{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 4 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) + 16$$
$$= \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$$

so repeated eigenvalue $\lambda_1 = \lambda_2 = -2$

for $\lambda = -2$, $(A - \lambda I)\vec{v} = \vec{0}$ is $\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $(A - \lambda I)\vec{u} = \vec{v}$ is $\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ take $\vec{u} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

and thus $\vec{y}_h(x) = c_1 e^{-2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2x}\left[x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right]$

$\vec{F}(x) = \begin{bmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{bmatrix} \Rightarrow \vec{y}_p(x) = \begin{bmatrix} a_1 x^2 + b_1 x + d_1 \\ a_2 x^2 + b_2 x + d_2 \end{bmatrix}$

so $\vec{y}_p{}'(x) = \begin{bmatrix} 2a_1 x + b_1 \\ 2a_2 x + b_2 \end{bmatrix}$

whereas $A\vec{y}_p + \vec{F} = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} a_1 x^2 + b_1 x + d_1 \\ a_2 x^2 + b_2 x + d_2 \end{bmatrix} + \begin{pmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{pmatrix}$

$$= \begin{bmatrix} (2a_1 - 4a_2 - 2)x^2 + (2b_1 - 4b_2 + 10)x + 2d_1 - 4d_2 \\ (4a_1 - 6a_2 - 4)x^2 + (4b_1 - 6b_2 + 12)x + 4d_1 - 6d_2 + 2 \end{bmatrix}$$

$$= \vec{y}_p{}' = \begin{bmatrix} 2a_1 x + b_1 \\ 2a_2 x + b_2 \end{bmatrix}$$

means that $\left.\begin{array}{l} 2a_1 - 4a_2 - 2 = 0 \\ 4a_1 - 6a_2 - 4 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_1 - 2a_2 = 1 \\ 2a_1 - 3a_2 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = 1 \\ a_2 = 0 \end{array}$

and $\left.\begin{array}{l} 2b_1 - 4b_2 + 10 = 2a_1 = 2 \\ 4b_1 - 6b_2 + 12 = 2a_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} b_1 - 2b_2 = -4 \\ 2b_1 - 3b_2 = -6 \end{array}\right\} \Rightarrow \begin{array}{l} b_1 = 0 \\ b_2 = 2 \end{array}$

and $\left.\begin{array}{l} 2d_1 - 4d_2 = b_1 = 0 \\ 4d_1 - 6d_2 + 2 = b_2 = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} d_1 - 2d_2 = 0 \\ 2d_1 - 3d_2 = 0 \end{array}\right\} \Rightarrow d_1 = d_2 = 0$

so $\vec{y}_p(x) = \begin{bmatrix} x^2 \\ 2x \end{bmatrix}$

and the general solution is

$$\vec{y}_g(x) = c_1 e^{-2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2x}\left(x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} x^2 \\ 2x \end{bmatrix}$$

$\vec{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \quad c_1 e^0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0\left(0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

or $\quad \left.\begin{array}{l} c_1 + \frac{1}{4} c_2 = 2 \\ c_1 \qquad\quad = 1 \end{array}\right\} \Rightarrow \quad c_2 = 4$

and the unique solution is

$$\boxed{\begin{aligned} \vec{y}(x) &= e^{-2x}\begin{bmatrix}1\\1\end{bmatrix} + 4e^{-2x}\left(x\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1/4\\0\end{bmatrix}\right) + \begin{bmatrix}x^2\\2x\end{bmatrix} \\ &= \begin{bmatrix}(4x+2)e^{-2x} + x^2 \\ (4x+1)e^{-2x} + 2x\end{bmatrix}\end{aligned}}$$

or $\boxed{y_1(x) = (4x+2)e^{-2x} + x^2 \quad \text{and} \quad y_2(x) = (4x+1)e^{-2x} + 2x}$.

A15

A6 5

3. $\int_0^1 \sec^2 x \, dx$ with GQ with $n=4$

change of variables $x = \frac{1}{2}((0)(1-t) + (1)(t+1)) = \frac{1}{2}(t+1)$

so $dx = \frac{1}{2} dt$

$$\int_0^1 \sec^2 x \, dx = \frac{1}{2}\int_{-1}^{1} \sec^2\left(\tfrac{1}{2}(t+1)\right) dt$$

$$\approx \frac{1}{2}\sum_{j=1}^{4} A_j \sec^2\left(\tfrac{1}{2}(t_j+1)\right)$$

$$= \frac{1}{2}\Big[A_1 \sec^2\left(\tfrac{1}{2}(t_1+1)\right) + A_2 \sec^2\left(\tfrac{1}{2}(t_2+1)\right) + A_3 \sec^2\left(\tfrac{1}{2}(t_3+1)\right) + A_4 \sec^2\left(\tfrac{1}{2}(t_4+1)\right)\Big]$$

$$= \frac{1}{2}\Big[0.3478548451 \sec^2\left(\tfrac{1}{2}(1-0.8611363116)\right)$$

$$+ 0.6521451549 \sec^2\left(\tfrac{1}{2}(1-0.3399810436)\right)$$

$$+ 0.6521451549 \sec^2\left(\tfrac{1}{2}(1+0.3399810436)\right)$$

$$+ 0.3478548451 \sec^2\left(\tfrac{1}{2}(1+0.8611363116)\right)\Big]$$

$$= \frac{1}{2}(0.3495371 + 0.7286616 + 1.0614589 + 0.9747627)$$

$$= \boxed{1.557210}$$

true value: $\int_0^1 \sec^2(x)\, dx = \tan(x)\Big|_0^1 = \tan(1) = 1.557408$

so the error is $\epsilon = \boxed{0.000198}$ (or 0.013%) (very good)