MAT 2384 3X Assignment # 4 Solutions

1. $y'' - 4y = 3e^{2x} + 12x - 2$, $y(0) = 9/2$, $y'(0) = -9/4$

the corresponding homog. DE is $y'' - 4y = 0$, which has characteristic equation $\lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$ and so $y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$

$r(x) = 3e^{2x} + 12x - 2 \Rightarrow y_p(x) = ax e^{2x} + bx + c$ (Mod Rule used)

then $y_p'(x) = ae^{2x} + 2axe^{2x} + b$

and $y_p''(x) = 4ae^{2x} + 4axe^{2x}$

so $y_p'' - 4y_p = 4ae^{2x} + 4axe^{2x} - 4(axe^{2x} + bx + c)$

$= 4ae^{2x} - 4bx - 4c = r(x) = 3e^{2x} + 12x - 2$

so $a = 3/4$, $b = -3$ and $c = 1/2$

thus $y_p(x) = \frac{3}{4} xe^{2x} - 3x + \frac{1}{2}$

then the general solution is $y_g(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{3}{4} xe^{2x} - 3x + \frac{1}{2}$

$y(0) = 9/2 \Rightarrow 9/2 = c_1 e^0 + c_2 e^0 + \frac{3}{4}(0)e^0 - 3(0) + \frac{1}{2} \Rightarrow c_1 + c_2 = 4$

$y_g'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} + \frac{3}{4}e^{2x} + \frac{3}{2} xe^{2x} - 3$

$y'(0) = -9/4 \Rightarrow -9/4 = 2c_1 e^0 - 2c_2 e^0 + \frac{3}{4}e^0 + \frac{3}{2}(0)e^0 - 3 \Rightarrow 2c_1 - 2c_2 = 0$

so $c_1 = c_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{2x} + 2e^{-2x} + \frac{3}{4} xe^{2x} - 3x + \frac{1}{2}}$

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2. $y'' - 5y' + 6y = 2e^x + 40\cos x$, $y(0) = y'(0) = 10$

corresp. homog DE is $y'' - 5y' + 6y = 0$, which has char. eq.

$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$

and so $y_h(x) = c_1 e^{2x} + c_2 e^{3x}$

$r(x) = 2e^x + 40\cos x \Rightarrow y_p(x) = ae^x + b\cos x + c\sin x$

then $y_p'(x) = ae^x - b\sin x + c\cos x$ and $y_p''(x) = ae^x - b\cos x - c\sin x$

so $y_p'' - 5y_p' + 6y_p = ae^x - b\cos x - c\sin x - 5(ae^x - b\sin x + c\cos x)$

$$+ 6(ae^x + b\cos x + c\sin x)$$

$$= 2ae^x + (5b - 5c)\cos x + (5c + 5b)\sin x$$

$$= r(x) = 2e^x + 40\cos x$$

so we have $a = 1$, $\left.\begin{matrix} 5b - 5c = 40 \\ 5b + 5c = 0 \end{matrix}\right\}$ $b = 4,\ c = -4$

and so $y_p(x) = e^x + 4\cos x - 4\sin x$

the general solution is $y_g(x) = c_1e^{2x} + c_2e^{3x} + e^x + 4\cos x - 4\sin x$

$\left.\begin{matrix} y(0) = 10 \Rightarrow 10 = c_1e^0 + c_2e^0 + e^0 + 4\cos(0) - 4\sin(0) \Rightarrow c_1 + c_2 = 5 \\ y_g'(x) = 2c_1e^{2x} + 3c_2e^{3x} + e^x - 4\sin x - 4\cos x \\ y'(0) = 10 \Rightarrow 10 = 2c_1e^0 + 3c_2e^0 + e^0 - 4\sin(0) - 4\cos(0) \Rightarrow 2c_1 + 3c_2 = 13 \end{matrix}\right\}$

so $c_2 = 2,\ c_3 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{2x} + 3e^{3x} + e^x + 4\cos x - 4\sin x}$

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3. $x^2y'' - 2xy' + 2y = x^2,\quad x > 0,\quad y(1) = 3,\quad y'(1) = 5$

correspond. homog. DE is $x^2y'' - 2xy' + 2y$, which has char. eq.

$$m(m-1) - 2m + 2 = m(m-1) - 2(m-1) = (m-1)(m-2) = 0$$

then $m_1 = 1,\ m_2 = 2$ and so $y_h(x) = c_1x + c_2x^2$

DE is EC $\Rightarrow$ must use Var of Parameters and $r(x) = 1$ (std form)

have to solve $\begin{matrix} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = r \end{matrix}$ or $\begin{matrix} u_1'x + u_2'x^2 = 0 \\ u_1' + 2u_2'x = 1 \end{matrix}$

or $\begin{matrix} u_1' + u_2'x = 0 & (1) \\ u_1' + 2u_2'x = 1 & (2) \end{matrix}$ $\quad (2) - (1) \Rightarrow u_2'x = 1 \Rightarrow u_2' = \frac{1}{x}$

so $u_2(x) = \ln x$

and then ① $\Rightarrow$ $u_1' = -u_2'x = -1$ $\Rightarrow$ $u_1(x) = -x$

so $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = (-x)(x) + (x^2)(\ln x)$

$= -x^2 + x^2 \ln x$ (but x^2 appears in $y_h(x)$)

so we'll take $y_p(x) = x^2 \ln x$

the general solution is $y_g(x) = c_1 x + c_2 x^2 + x^2 \ln x$

$y(1) = 3 \Rightarrow 3 = c_1(1) + c_2(1)^2 + (1)^2 \ln(1) \Rightarrow c_1 + c_2 = 3$

$y_g'(x) = c_1 + 2c_2 x + 2x\ln x + x$

$y'(1) = 5 \Rightarrow 5 = c_1 + 2c_2(1) + 2(1)\ln(1) + 1 \Rightarrow c_1 + 2c_2 = 4$

$\left.\right\} c_1 = 2, \; c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = 2x + x^2 + x^2 \ln x}$

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4. $y'' - 4y' + 4y = 3x^{-2}e^{2x}$, $y(1) = y'(1) = 2e^2$

corresp'd homog DE is $y'' - 4y' + 4y = 0$, which has char. eq.

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$ and so $y_h(x) = c_1 e^{2x} + c_2 x e^{2x}$

$r(x) = 3x^{-2}e^{2x} \Rightarrow$ must use Var of Par's

$u_1'y_1 + u_2'y_2 = 0$ is $u_1' e^{2x} + u_2' x e^{2x} = 0$

$u_1'y_1' + u_2'y_2' = r$ $\quad 2u_1' e^{2x} + u_2'(e^{2x} + 2xe^{2x}) = 3x^{-2}e^{2x}$

or $u_1' + u_2' x = 0$ ① $\quad$ ② $- 2\times$① $\Rightarrow$ $u_2' = 3x^{-2}$

$2u_1' + u_2'(1 + 2x) = 3x^{-2}$ ② $\quad$ so $u_2(x) = -3x^{-1}$

then ① $\Rightarrow$ $u_1' = -u_2'x = -3x^{-1}$ $\Rightarrow$ $u_1(x) = -3\ln|x|$

so $y_p(x) = (-3\ln|x|)(e^{2x}) + (-3x^{-1})(xe^{2x})$

$= -3e^{2x}\ln|x| - 3e^{2x}$ $\leftarrow$ appears in $y_h(x)$

so take $y_p(x) = -3e^{2x}\ln|x|$

the general solution is $y_g(x) = c_1 e^{2x} + c_2 x e^{2x} - 3e^{2x}\ln|x|$

$y(1) = 2e^2 \Rightarrow \quad 2e^2 = c_1 e^2 + c_2(1)e^2 - 3e^2 \ln(1) \Rightarrow \quad c_1 + c_2 = 2$

$y_g'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} - 6e^{2x}\ln|x| - 3x^{-1}e^{2x}$

$y'(1) = 2e^2 \Rightarrow \quad 2e^2 = 2c_1 e^2 + c_2 e^2 + 2c_2 e^2 - 6e^2 \ln(1) - 3(1)^{-1}e^2$

or $\quad 2c_1 + 3c_2 = 5 \qquad$ so $\quad c_1 = c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = e^{2x} + xe^{2x} - 3e^{2x}\ln|x|}$

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5. $\quad y''' - 4y'' + y' + 6y = 8\cos x + 18x + 3, \quad y(0) = -2, y'(0) = -11, y''(0) = -34$

correspd homog DE is $y''' - 4y'' + y' + 6y = 0$, which has. char. eq.

$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$ by inspection, $\lambda = -1$ is a root, so

$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda+1)(\lambda^2 - 5\lambda + 6) = (\lambda+1)(\lambda-2)(\lambda-3)$

hence $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3$ and $\quad y_h(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$

$r(x) = 8\cos x + 18x + 3 \Rightarrow y_p(x) = a\cos x + b\sin x + dx + f$

thus $y_p'(x) = -a\sin x + b\cos x + d, \quad y_p''(x) = -a\cos x - b\sin x$

and $y_p'''(x) = a\sin x - b\cos x$

so $y_p''' - 4y_p'' + y_p' + 6y_p = a\sin x - b\cos x - 4(-a\cos x - b\sin x)$

$\qquad - a\sin x + b\cos x + d + 6(a\cos x + b\sin x + dx + f)$

$\qquad = 10a\cos x + 10b\sin x + 6dx + d + 6f$

$\qquad = r(x) = 8\cos x + 18x + 3$

thus $\quad a = 4/5, \quad b = 0, \quad d = 3, \quad f = 0$

so $\quad y_p(x) = \frac{4}{5}\cos x + 3x$

and the general solution is $y_g(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x} + \frac{4}{5}\cos x + 3x$

$y(0) = -2 \Rightarrow \quad -2 = c_1 e^0 + c_2 e^0 + c_3 e^0 + \frac{4}{5}\cos(0) + 3(0) \Rightarrow \quad c_1 + c_2 + c_3 = -14/5$ ①

$y_g'(x) = -c_1 e^{-x} + 2c_2 e^{2x} + 3c_3 e^{3x} - \frac{4}{5}\sin x + 3$

$y'(0) = -11 \Rightarrow \quad -11 = -c_1 e^0 + 2c_2 e^0 + 3c_3 e^0 - \frac{4}{5}\sin(0) + 3 \Rightarrow -c_1 + 2c_2 + 3c_3 = -14$ ②

$y_g''(x) = c_1 e^{-x} + 4c_2 e^{2x} + 9c_3 e^{3x} - \frac{4}{5}\cos x$

$y''(0) = -34 \Rightarrow \quad -34 = c_1 e^0 + 4c_2 e^0 + 9c_2 e^0 - \frac{4}{5}\cos(0) \Rightarrow \quad c_1 + 4c_2 + 9c_3 = \frac{-166}{5}$ ③

$$① + ② \Rightarrow \quad 3c_2 + 4c_3 = -84/5 \quad ④ \qquad ⑤ - 2\times④ \Rightarrow 4c_3 = -68/5$$
$$② + ③ \Rightarrow \quad 6c_2 + 12c_3 = -236/5 \quad ⑤ \qquad c_3 = -17/5$$

then $c_2 = \frac{1}{3}\left(-\frac{84}{5} + \frac{68}{5}\right) = -16/15$

and $c_1 = -\frac{14}{5} + \frac{16}{15} + \frac{17}{5} = 25/15 = 5/3$

$\therefore$ the unique solution is $\boxed{y(x) = \frac{5}{3}e^{-x} - \frac{16}{15}e^{2x} - \frac{17}{5}e^{3x} + \frac{4}{5}\cos x + 3x}$

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6. $\int_0^4 \frac{2x}{1+x^2}\,dx$ with Simpson with $2n = 8$

then $h = \frac{4-0}{8} = 1/2 \Rightarrow x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2,$
$x_5 = 2.5,\ x_6 = 3,\ x_7 = 3.5,\ x_8 = 4$

$$\int_0^4 \frac{2x}{1+x^2}\,dx \approx \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left[\frac{2x_0}{1+x_0^2} + 4\left(\frac{2x_1}{1+x_1^2}\right) + 2\left(\frac{2x_2}{1+x_2^2}\right) + 4\left(\frac{2x_3}{1+x_3^2}\right)\right.$$
$$\left. + 2\left(\frac{2x_4}{1+x_4^2}\right) + 4\left(\frac{2x_5}{1+x_5^2}\right) + 2\left(\frac{2x_6}{1+x_6^2}\right) + 4\left(\frac{2x_7}{1+x_7^2}\right) + \frac{2x_8}{1+x_8^2}\right]$$
$$= \frac{1}{6}\left[0 + 4\left(\frac{1}{1+(0.5)^2}\right) + 2\left(\frac{2}{1+1^2}\right) + 4\left(\frac{3}{1+1.5^2}\right) + 2\left(\frac{4}{1+2^2}\right) + 4\left(\frac{5}{1+2.5^2}\right)\right.$$
$$\left. + 2\left(\frac{6}{1+3^2}\right) + 4\left(\frac{7}{1+3.5^2}\right) + \frac{8}{1+4^2}\right]$$
$$= \frac{1}{6}(0 + 3.2 + 2 + 3.6923077 + 1.6 + 2.7586207$$
$$+ 1.2 + 2.1132075 + 0.4705882)$$

$$= \boxed{2.839127}$$

true value: $\int_0^4 \frac{2x}{1+x^2}\,dx = \ln(1+x^2)\Big|_0^4 = \ln(17) = 2.833213$

so the error is $\epsilon = \boxed{0.005908}$ (or 0.21%) (very good)