MAT 2384 3X Assignment # 3 : Solutions

1. $y''' - 6y'' + 11y' - 6y = 0, \quad y(0)=6,\ y'(0)=14,\ y''(0)=36$

the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$
by inspection, $\lambda = 1$ is a root, so $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda^2 - 5\lambda + 6)$
so $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ $\quad = (\lambda-1)(\lambda-2)(\lambda-3)$

the general solution is $y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

$y(0) = 6 \Rightarrow 6 = c_1 e^0 + c_2 e^0 + c_3 e^0 \Rightarrow c_1 + c_2 + c_3 = 6$ ①
$y'(x) = c_1 e^{x} + 2c_2 e^{2x} + 3c_3 e^{3x}$
$y'(0) = 14 \Rightarrow 14 = c_1 e^0 + 2c_2 e^0 + 3c_3 e^0 \Rightarrow c_1 + 2c_2 + 3c_3 = 14$ ②
$y''(x) = c_1 e^{x} + 4c_2 e^{2x} + 9c_3 e^{3x}$
$y''(0) = 36 \Rightarrow 36 = c_1 e^0 + 4c_2 e^0 + 9c_3 e^0 \Rightarrow c_1 + 4c_2 + 9c_3 = 36$ ③

③ − ① $\quad 3c_2 + 8c_3 = 30$ ④
② − ① $\quad c_2 + 2c_3 = 8$ ⑤

④ − 3×⑤ $\Rightarrow 2c_3 = 6 \Rightarrow c_3 = 3$
then $c_2 = 2$ and $c_1 = 1$

∴ the unique solution is $\boxed{y(x) = e^{x} + 2e^{2x} + 3e^{3x}}$

2. $y^{(4)} + 5y'' + 4y = 0, \quad y(0) = -1,\ y'(0) = 2,\ y''(0) = 4,\ y'''(0) = -2$

the char. eq. is $\lambda^4 + 5\lambda^2 + 4 = (\lambda^2 + 1)(\lambda^2 + 4) = 0$
so $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \pm 2i$
and the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos(2x) + c_4 \sin(2x)$$

$y(0) = -1 \Rightarrow -1 = c_1 \cos(0) + c_2 \sin(0) + c_3 \cos(0) + c_4 \sin(0)$
$\Rightarrow c_1 + c_3 = -1$ ①

$y'(x) = -c_1 \sin x + c_2 \cos x - 2c_3 \sin(2x) + 2c_4 \cos(2x)$

$y'(0) = 2 \Rightarrow 2 = -c_1 \sin(0) + c_2 \cos(0) - 2c_3 \sin(0) + 2c_4 \cos(0)$

$\Rightarrow c_2 + 2c_4 = 2$ (2)

$y''(x) = -c_1 \cos x - c_2 \sin x - 4c_3 \cos(2x) - 4c_4 \sin(2x)$

$y''(0) = 4 \Rightarrow 4 = -c_1 \cos(0) - c_2 \sin(0) - 4c_3 \cos(0) - 4c_4 \sin(0)$

$\Rightarrow -c_1 - 4c_3 = 4$ (3)

$y'''(x) = c_1 \sin x - c_2 \cos x + 8c_3 \sin(2x) - 8c_4 \cos(2x)$

$y'''(0) = -2 \Rightarrow -2 = c_1 \sin(0) - c_2 \cos(0) + 8c_3 \sin(0) - 8c_4 \cos(0)$

$\Rightarrow -c_2 - 8c_4 = -2$ (4)

(1) + (3) $\Rightarrow -3c_3 = 3 \Rightarrow c_3 = -1 \Rightarrow c_1 = 0$

(2) + (4) $\Rightarrow -6c_4 = 0 \Rightarrow c_4 = 0 \Rightarrow c_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2\sin x - \cos(2x)}$

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3. $x^3 y''' - x^2 y'' + 2xy' - 2y = 0$, $x > 0$, $y(1) = -3$, $y'(1) = -7$, $y''(1) = -9$

the characteristic equation is $m(m-1)(m-2) - m(m-1) + 2m - 2 = 0$

or $m(m-1)(m-2) - m(m-1) + 2(m-1) = 0$

or $(m-1)\left(m(m-2) - (m-2)\right) = 0$

so $(m-1)^2(m-2) = 0 \Rightarrow m_{1,2} = 1,\ m_3 = 2$

so the general solution is $y(x) = c_1 x + c_2 x \ln x + c_3 x^2$

$y(1) = -3 \Rightarrow -3 = c_1(1) + c_2(1)\ln(1) + c_3(1)^2 \Rightarrow c_1 + c_3 = -3$ (1)

$y'(x) = c_1 + c_2 \ln x + c_2 + 2c_3 x$

$y'(1) = -7 \Rightarrow -7 = c_1 + c_2 \ln(1) + c_2 + 2c_3(1) \Rightarrow c_1 + c_2 + 2c_3 = -7$ (2)

$y''(x) = c_2/x + 2c_3$

$y''(1) = -9 \Rightarrow -9 = c_2/(1) + 2c_3 \Rightarrow c_2 + 2c_3 = -9$ (3)

3 − 3 $\Rightarrow$ $c_1 = 2$ $\Rightarrow$ $c_3 = -5$ $\Rightarrow$ $c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = 2x + x\ln x - 5x^2}$

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4. $h = \Delta x = 1$

the formulas are
$$f_0' \approx \frac{1}{2h}\left[-3f_0 + 4f_1 - f_2\right]$$
$$f_1' \approx \frac{1}{2h}\left[-f_0 + f_2\right]$$
$$f_2' \approx \frac{1}{2h}\left[f_0 - 4f_1 + 3f_2\right]$$

then
$$f'(1.5) \simeq \frac{1}{2}\left(-3(4.3) + 4(6.7) - (7.4)\right) = \boxed{3.25}$$
$$f'(2.5) \simeq \frac{1}{2}\left(-(4.3) + (7.4)\right) = \boxed{1.55}$$
$$f'(3.5) \simeq \frac{1}{2}\left((4.3) - 4(6.7) + 3(7.4)\right) = \boxed{-0.15}$$