MAT 2384 3X Assignment #3 Solutions

1. $y'' - 2y = 0$, $y(0) = 2$, $y'(0) = 0$

the characteristic equation is $\lambda^2 - 2 = (\lambda - \sqrt{2})(\lambda + \sqrt{2}) = 0$

so the roots are $\lambda_{1,2} = \pm\sqrt{2}$

and the general solution is $y(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$

$$\left.\begin{array}{l} y(0) = 2 \Rightarrow 2 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 2 \\ y'(x) = \sqrt{2} c_1 e^{\sqrt{2}x} - \sqrt{2} c_2 e^{-\sqrt{2}x} \\ y'(0) = 0 \Rightarrow 0 = \sqrt{2} c_1 e^0 - \sqrt{2} c_2 e^0 \Rightarrow c_1 - c_2 = 0 \end{array}\right\} c_1 = c_2 = 1$$

$\therefore$ the unique solution is $\boxed{y(x) = e^{\sqrt{2}x} + e^{-\sqrt{2}x}}$

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2. $y'' + \pi^2 y = 0$, $y(0) = 0$, $y'(0) = \pi$

the char. eq. is $\lambda^2 + \pi = 0 \Rightarrow$ the roots are $\lambda_{1,2} = \pm\pi i$

and the general solution is $y(x) = c_1 \cos(\pi x) + c_2 \sin(\pi x)$

$y(0) = 0 \Rightarrow 0 = c_1 \cos(0) + c_2 \sin(0) \Rightarrow c_1 = 0$

$y'(x) = -\pi c_1 \sin(\pi x) + \pi c_2 \cos(\pi x)$

$y'(0) = \pi \Rightarrow \pi = -\pi c_1 \sin(0) + \pi c_2 \cos(0) \Rightarrow c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = \sin(\pi x)}$

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3. $y'' - 4y' + 20y = 0$, $y(0) = 2$, $y'(0) = 0$

the char. eq. is $\lambda^2 - 4\lambda + 20 = 0$

the roots are $\lambda_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4(20)}}{2} = \dfrac{4 \pm \sqrt{-64}}{2} = 2 \pm 4i$

the general solution is $y(x) = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(4x)$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) \Rightarrow c_1 = 2$

$y'(x) = 2c_1 e^{2x} \cos(4x) - 4c_1 e^{2x} \sin(4x) + 2c_2 e^{2x} \sin(4x)$
$\quad + 4c_2 e^{2x} \cos(4x)$

$y'(0) = 0 \Rightarrow \quad 0 = 2c_1 e^0 \cos(0) - 4c_1 e^0 \sin(0)$

$+ 2c_2 e^0 \sin(0) + 4c_2 e^0 \cos(0)$

$= 2c_1 + 4c_2 \Rightarrow c_2 = -1$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^{2x}\cos(4x) - e^{2x}\sin(4x)}$

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4. $y'' + 6y' + 9y = 0, \quad y(0) = 7, \quad y'(0) = -13$

the char. eq. is $\lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0 \Rightarrow$ roots are $\lambda_1 = \lambda_2 = -3$

and the general solution is $y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

$y(0) = 7 \Rightarrow \quad 7 = c_1 e^0 + c_2(0)e^0 \Rightarrow c_1 = 7$

$y'(x) = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$

$\Rightarrow c_2 = 8$

$y'(0) = -13 \Rightarrow \quad -13 = \underbrace{-3c_1 e^0 + c_2 e^0 - 3c_2(0)e^0} = -3c_1 + c_2$

$\therefore$ the unique solution is $\boxed{y(x) = 7e^{-3x} + 8xe^{-3x}}$

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5. $x^2y'' - 4xy' + 6y = 0, \quad y(1) = 4, \quad y'(1) = 10$

the characteristic equation is $m(m-1) - 4m + 6 = m^2 - 5m + 6 = 0$

or $(m-2)(m-3) = 0$, so the roots are $m_1 = 2, m_2 = 3$

and the general solution is $y(x) = c_1 x^2 + c_2 x^3$

$y(1) = 4 \Rightarrow \quad 4 = c_1(1)^2 + c_2(1)^3 \Rightarrow \quad c_1 + c_2 = 4$

$y'(x) = 2c_1 x_1 + 3c_2 x^2$

$y'(1) = 10 \Rightarrow \quad 10 = 2c_1(1) + 3c_2(1)^2 \Rightarrow \quad 2c_1 + 3c_2 = 10$

$\left.\right\} c_1 = c_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2x^2 + 2x^3}$

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6. $x^2y'' - 5xy' + 13y = 0, \quad x > 0, \quad y(1) = 0, \quad y'(1) = 6$

Char. eq. is $m(m-1) - 5m + 13 = m^2 - 6m + 13 = 0$

so the roots are $m_{1,2} = \dfrac{6 \pm \sqrt{(-6)^2 - 4(13)}}{2} = \dfrac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$

and the general solution is $y(x) = C_1 x^3 \cos(2\ln x) + C_2 x^3 \sin(2\ln x)$

$y(1) = 0 \Rightarrow \quad 0 = C_1 (1)^3 \cos(0) + C_2 (1)^3 \sin(0) \Rightarrow C_1 = 0$

$y'(x) = 3C_1 x^2 \cos(2\ln x) - 2C_1 x^2 \sin(2\ln x) + 3C_2 x^2 \sin(2\ln x)$
$\quad + 2C_2 x^2 \cos(2\ln x)$

$y'(1) = 6 \Rightarrow \quad 6 = 3C_1 (1)^2 \cos(0) - 2C_1 (1)^2 \sin(0) \quad \Rightarrow C_2 = 3$
$\quad + 3C_2 (1)^2 \sin(0) + 2C_2 (1)^2 \cos(0) = 2C_2$

$\therefore$ the unique solution is $\boxed{y(x) = 3x^3 \sin(2\ln x)}$

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7. $x^2 y'' + 5xy' + 4y = 0, \quad x > 0, \quad y(1) = 4, \quad y'(1) = 0$

char. eq. is $m(m-1) + 5m + 4 = m^2 + 4m + 4 = (m+2)^2 = 0$

so $m_1 = m_2 = -2$ and the general solution is $y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x$

$y(1) = 4 \Rightarrow \quad 4 = C_1 (1)^{-2} + C_2 (1)^{-2} \ln(1) \Rightarrow C_1 = 4$

$y'(x) = -2C_1 x^{-3} - 2C_2 x^{-3} \ln x + C_2 x^{-3}$

$y'(1) = 0 \Rightarrow \quad 0 = -2C_1 (1)^{-3} - 2C_2 (1)^{-3} \ln(1) + C_2 (1)^{-3}$
$\quad = -2C_1 + C_2 \Rightarrow C_2 = 8$

$\therefore$ the unique solution is $y(x) = \boxed{4x^{-2} + 8x^{-2} \ln x}$

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8. $(x_0, f_0) = (1.2, 2.7), \ (x_1, f_1) = (1.6, 5.1), \ (x_2, f_2) = (2.1, 8.7)$ and $(x_3, f_3) = (2.7, 12$

$p_3(x) = \sum_{j=0}^{3} L_j(x) f_j = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3$

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 1.6)(x - 2.1)(x - 2.7)}{(1.2 - 1.6)(1.2 - 2.1)(1.2 - 2.7)}$$

$$= \frac{(x - 1.6)(x^2 - 4.8x + 5.67)}{(-0.4)(-0.9)(-1.5)} = \frac{x^3 - 6.4x^2 + 13.35x - 9.072}{-0.54}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 1.2)(x - 2.1)(x - 2.7)}{(1.6 - 1.2)(1.6 - 2.1)(1.6 - 2.7)}$$

$$= \frac{(x-1.2)(x^2-4.8x+5.67)}{(0.4)(-0.5)(-1.1)} = \frac{x^3-6x^2+11.43x-6.804}{0.22}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-1.2)(x-1.6)(x-2.7)}{(2.1-1.2)(2.1-1.6)(2.1-2.7)}$$

$$= \frac{(x-1.2)(x^2-4.3x+4.32)}{(0.9)(0.5)(-0.6)} = \frac{x^3-5.5x^2+9.48x-5.184}{-0.27}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-1.2)(x-1.6)(x-2.1)}{(2.7-1.2)(2.7-1.6)(2.7-2.1)}$$

$$= \frac{(x-1.2)(x^2-3.7x+3.36)}{(1.5)(1.1)(0.6)} = \frac{x^3-4.9x^2+7.8x-4.032}{0.99}$$

so $p_3(x) = L_0(x)(2.7) + L_1(x)(5.1) + L_2(x)(8.7) + L_3(10.3)$

$$= \boxed{-3.6364x^3 + 19.1515x^2 - 26.0970x + 12.7218}$$

(check: $p_3(1.2) \simeq 2.7$, $p_3(1.6) \simeq 5.1$, $p_3(2.1) \simeq 8.7$, $p_3(2.7) \simeq 10.3$ okay!)

$f(1.5) \simeq p_3(1.5) = \boxed{4.3943}$ $\quad \epsilon_{min} = \boxed{0.0005}$ $\quad \epsilon_{max} = \boxed{0.0036}$

$f(2) \simeq p_3(2) = \boxed{8.0426}$ $\quad \epsilon_{min} = \boxed{0.0005}$ $\quad \epsilon_{max} = \boxed{0.0036}$

$$|\epsilon_3(x)| = \left|(x-x_0)(x-x_1)(x-x_2)(x-x_3)\frac{f^{(4)}(t)}{4!}\right|$$

$$= |(x-1.2)(x-1.6)(x-2.1)(x-2.7)f^{(4)}(t)/24|$$

so $|\epsilon_3(1.5)| = |(1.5-1.2)(1.5-1.6)(1.5-2.1)(1.5-2.7)f^{(4)}(t)/24| = 0.0009\, f^{(4)}(t)$

and $|\epsilon_3(2)| = |(2-1.2)(2-1.6)(2-2.1)(2-2.7)f^{(4)}(t)/24| = 0.0009\, f^{(4)}(t)$