MAT 2384 C Assignment #1 Solutions

1. $2(1+x^2)yy' = 2x+3$, $y(0)=1$

the DE is separable : $2yy' = \dfrac{2x+3}{1+x^2}$

so $\int 2y\,dy = \int \left(\dfrac{2x}{1+x^2} + \dfrac{3}{1+x^2}\right)dx + C$

giving $y^2 = \ln(1+x^2) + 3\arctan x + C$ (general solution)

$y(0)=1$ (and $y>0$) means that

$(1)^2 = \ln(1) + 3\arctan(0) + C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = \sqrt{\ln(1+x^2) + 3\arctan x + 1}}$

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2. $\cot x \sin y\, y' = 1$, $y(0)=0$

this DE is also separable : $\sin y\, y' = \tan x$

and so $\int \sin y\,dy = \int \dfrac{\sin x}{\cos x}\,dx + C$

which gives $-\cos y = -\ln|\cos x| + C$

or $\cos y = \ln|\cos x| + C$

and so the general solution is $y = \arccos(\ln|\cos x| + C)$

$y(0)=0 \Rightarrow 0 = \arccos(\ln(\cos(0)) + C)$

$0 = \arccos(\ln(1) + C) = \arccos(C)$

thus $C = 1$. and the unique solution is

$\boxed{y = \arccos(\ln|\cos x| + 1)}$

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3. $(x\tan(y/x) - y)\,dx + x\,dy = 0$, $y(1) = \pi/2$

$\left.\begin{aligned} M(x,y) &= x\tan(y/x) - y \\ N(x,y) &= x \end{aligned}\right\}$ both homogeneous of degree 1

we let $u = y/x$, ie $y = ux$, $dy = u\,dx + x\,du$

and the DE becomes

$(x\tan u - ux)\,dx + x(u\,dx + x\,du) = 0$

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or $x \tan u \, dx - \cancel{ux \, dx} + \cancel{ux \, dx} + x^2 du = 0$

or $x \tan u \, dx + x^2 du = 0$

separates as $\cot u \, du = -\frac{1}{x} dx$

integrate $\int \frac{\cos u}{\sin u} du = \int -\frac{1}{x} dx + C$

we get $\ln|\sin u| = -\ln|x| + C$

or $\sin u = e^{-\ln|x| + C} = e^C e^{-\ln|x|} = K/x$

thus $u = \arcsin(K/x) \Rightarrow y = x \arcsin(K/x)$ (gen. sol'n)

$y(1) = \pi/2 \Rightarrow \pi/2 = (1)\arcsin(K/1) = \arcsin(K)$

then $K = 1$ and the unique solution

$$\boxed{y = x \arcsin(1/x)}$$

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4. $(4xy^4 + 3y^2)\,dx + (8x^2y^3 + 6xy + 3y^2)\,dy = 0, \quad y(1) = -1$

$M(x,y) = 4xy^4 + 3y^2 \Rightarrow M_y = 16xy^3 + 6y$

$N(x,y) = 8x^2y^3 + 6xy + 3y^2 \Rightarrow N_x = 16xy^3 + 6y$

$\Big\}$ $M_y = N_x$ DE is exact

then $F(x,y) = \int M(x,y)\,dx + g(y)$ (or $\int N(x,y)\,dy + g(x)$)

$= \int (4xy^4 + 3y^2)\,dx + g(y)$

$= 2x^2y^4 + 3xy^2 + g(y)$

so $\frac{\partial F}{\partial y} = 8x^2y^3 + 6xy + g'(y) = N(x,y) = 8x^2y^3 + 6xy + 3y^2$

thus $g'(y) = 3y^2 \Rightarrow g(y) = y^3$

then $F(x,y) = 2x^2y^4 + 3xy^2 + y^3$

and the general solution is $2x^2y^4 + 3xy^2 + y^3 = C$

$y(1) = -1 \Rightarrow 2(1)^2(-1)^4 + 3(1)(-1)^2 + (-1)^3 = C \Rightarrow C = 4$

$\therefore$ the unique solution is $\boxed{2x^2y^4 + 3xy^2 + y^3 = 4}$

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5. $\left(e^y + \frac{3}{x}\right)dx + (xe^y - 4\sin y)\,dy = 0, \quad y(1) = 0$

$M(x,y) = e^y + \frac{3}{x} \Rightarrow M_y = e^y$
$N(x,y) = xe^y - 4\sin y \Rightarrow N_x = e^y$
$\Big\}$ $M_y = N_x$ DE is exact

$F(x,y) = \int N(x,y)\,dy + g(x) \quad \left(\text{or } \int M(x,y)\,dx + g(y)\right)$
$= \int (xe^y - 4\sin y)\,dy + g(x)$
$= xe^y + 4\cos y + g(x)$

so $\frac{\partial F}{\partial x} = e^y + g'(x) = M(x,y) = e^y + \frac{3}{x}$

and so $g'(x) = \frac{3}{x} \Rightarrow g(x) = 3\ln|x|$

thus $F(x,y) = xe^y + 4\cos y + 3\ln|x|$

and the general solution is $xe^y + 4\cos y + 3\ln|x| = C$

$y(1) = 0 \Rightarrow (1)e^0 + 4\cos(0) + 3\ln(1) = C \Rightarrow C = 5$

$\therefore$ the unique solution is $\boxed{xe^y + 4\cos y + 3\ln|x| = 5}$

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6. $f(x) = x^3 - 6x + 4$ $\quad f(0) = 4$ $\quad \therefore$ sign change on $[0,1]$
$f(1) = -1$ $\quad$ so root in $[0,1]$

$f(x) = x^3 - 6x + 4 = 0 \Rightarrow x = \frac{x^3+4}{6}$

take $g(x) = \frac{x^3+4}{6}$, which will satisfy the fixed-pt condition

also $|g'(x)| = \left|\frac{3x^2}{6}\right| = \frac{1}{2}x^2 \le \frac{1}{2} < 1$ on $[0,1]$

then $x_{n+1} = g(x_n)$ will generate a convergent sequence

$x_0 = 0.5, \quad x_1 = g(0.5) = \frac{(0.5)^3+4}{6} = 0.6875$

$x_2 = g(0.6875) = 0.720825$
$x_3 = g(0.720825) = 0.729089$
$x_4 = g(0.729089) = 0.731260$

$x_5 = g(0.731\,260) = 0.731\,839$
$x_6 = g(0.731\,839) = 0.731\,994$
$x_7 = g(0.731\,994) = 0.732\,036$
$x_8 = g(0.732\,036) = 0.732\,047$
$x_9 = g(0.732\,047) = 0.732\,050$
$x_{10} = g(0.732\,050) = 0.732\,051$
$x_{11} = g(0.732\,051) = 0.732\,051$
$\therefore$ stop

(check: $f(0.732\,051) \simeq -8.45 \times 10^{-7}$ $\therefore$ okay!)
$\therefore$ the root is $\boxed{0.732\,051}$

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7. $f(x) = x^3 - 6x + 4$, $f'(x) = 3x^2 - 6$, $x_0 = 0.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 6x_n + 4}{3x_n^2 - 6} = \frac{2x_n^3 - 4}{3x_n^2 - 6}$$

$x_0 = 0.5$, $x_1 = \dfrac{2(0.5)^3 - 4}{3(0.5)^2 - 6} = 0.714\,286$

$x_2 = 0.731\,898$
$x_3 = 0.732\,051$
$x_4 = 0.732051$ $\therefore$ stop root is $\boxed{0.732\,051}$

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8. if $\ln x = 9 - x^2$, then $f(x) = \ln x - 9 + x^2 = 0$
so $f'(x) = \frac{1}{x} + 2x$

so $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n) - 9 + x_n^2}{\frac{1}{x_n} + 2x_n} = \frac{10 - \ln x_n + x_n^2}{\frac{1}{x_n} + 2x_n}$$

$x_0 = 2$, $x_1 = \dfrac{10 - \ln(2) + (2)^2}{\frac{1}{2} + 2(2)} = 2.9571$

$x_2 = 2.8246$
$x_3 = 2.8218$
$x_4 = 2.8218$ $\therefore$ stop

(check: $\ln(2.8218) = 1.037375$
$9 - (2.8218)^2 = 1.0374448$
$\therefore$ okay!)

$\therefore$ the solution is $\boxed{x = 2.8218}$