University of Ottawa
Dept. of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 3X - Spring-Summer 2017
Midterm II - V.2
Professor: Abdelkrim El basraoui
Duration: 80 minutes.

Solutions

Name:__________________ ID Number:__________________

Instructions: (Please read carefully.)

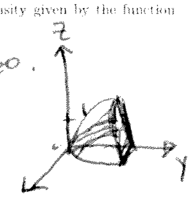
- This exam has 9 pages and 5 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1 and 2 are worth 5 marks each, Questions 3 and 4 are worth 4 marks each, and Question 5 is worth 7 marks, so organize your time accordingly.
- Answer each question in the space provided or using backs of pages if necessary.
- Page 8 is an extra page for Question 5.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
- Good luck!
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ____________________________

Question	Q 1	Q 2	Q 3	Q 4	Q 5	Total
Maximum	5	5	4	4	7	25
Score						

1. Consider the solid in the first octant bounded by the planes $z = y$, $z = 0$ and $y = 1$ and the parabolic cylinder $y = x^2$. This solid has a mass density given by the function $\delta(x, y, z) = x$. Find the total mass of this solid.

- In the first octant $x \geq 0, y \geq 0, z \geq 0$.
- The region of integration is



$E = \{(x,y,z) \mid (x,y) \text{ in } D \,;\, 0 \leq z \leq y\}$

where D is the projection of E in the xy-plane.

So $D = \{(x,y) \mid 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1\}$

or $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

- The mass is then

$$m = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y x \, dz\, dx\, dy \overset{\text{or}}{=} \int_0^1 \int_0^{x^2} \int_0^y x \, dz\, dy\, dx$$

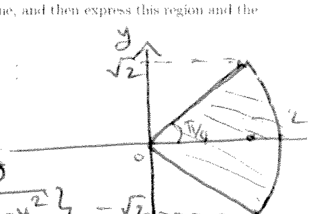
$$\boxed{m = \frac{1}{12}}.$$

2. Compute the following double integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{|y|}^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$$

Hint: Sketch the region of integration in the xy-plane, and then express this region and the integral in a different coordinate system

- Region of integration



$D = \{(x,y) \mid -\sqrt{2} \leq y \leq \sqrt{2},\ |y| \leq x \leq \sqrt{4-y^2}\}$.

- In polar coords, we have

$D = \{(r,\theta) \mid 0 \leq r \leq 2,\ -\pi/4 \leq \theta \leq \pi/4\}$.

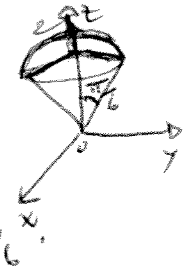
& $x^2 + y^2 = r^2$. ($dx\,dy = r\,dr\,d\theta$)

Also, the integral in polar coords is

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{|y|}^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy = \int_0^2 \int_{-\pi/4}^{\pi/4} r^2\, r\, d\theta\, dr$$

$$= \boxed{2\pi}$$

3. Consider the solid in the first octant which is bounded by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=4$. This solid has a mass density given by the function $\delta(x,y,z) = 18 - 3x^2 - 3y^2 - 3z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



- The angle is making the cone with the z-axis satisfies $\tan(\phi) = \frac{1}{\sqrt{3}} \Rightarrow \phi = \pi/6$, since $0 \le \phi \le \pi$.
- So, the inclination is $0 \le \phi \le \pi/6$.
- In the 1st octant, $0 \le \theta \le \pi/2$.
- The radius of the sphere is 2, & so $0 \le \rho \le 2$.
- In spherical coords. $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ & $\delta = 18 - 3\rho^2$.
- The total mass is:

$$\boxed{m = \int_0^2 \int_0^{\pi/2} \int_0^{\pi/6} (18-3\rho^2)\, \rho^2 \sin(\phi)\, d\phi \, d\theta \, d\rho.}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \cos(t)\,\vec{\mathbf{i}} + \sin(t)\,\vec{\mathbf{j}} + 3t\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/4.$$

- $\vec{r}\,'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j} + 3\vec{k}$.
- $|\vec{r}\,'(t)| = \sqrt{10}$.
- Arclength:

$$L = \int_0^{\pi/4} |\vec{r}\,'(t)|\, dt = \boxed{\frac{\pi\sqrt{10}}{4}}$$

5. Among the following three vector fields there is **exactly one** conservative vector field.

$$\vec{F} = x^2y\,\vec{\mathbf{i}} - xy^2\,\vec{\mathbf{j}}$$
$$\vec{G} = 2xe^{-y}\,\vec{\mathbf{i}} + (2y - x^2e^{-y})\,\vec{\mathbf{j}}$$
$$\vec{H} = e^x\cos(xy)\,\vec{\mathbf{i}} + e^x\sin(xy)\,\vec{\mathbf{j}}$$

(a) Determine which **one** of these vector fields is conservative. If a vector field is not conservative, justify why it is not conservative.

(b) Find a potential function f **for the conservative vector field** in (a).

(c) Evaluate the line integral of **the conservative vector field** in (a) along the curve C, where C is any path form $(0,0)$ to $(1,1)$.

(a) • $\vec{F}$? : $\frac{\partial}{\partial y}(x^2y) = x^2 \ne \frac{\partial}{\partial x}(-xy^2) = -y^2$

$\Rightarrow$ $\vec{F}$ is not conservative.

• $\vec{G}$? : $\frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y} = \frac{\partial}{\partial x}(2y - x^2e^{-y})$

$\Rightarrow$ $\boxed{\vec{G} \text{ is conservative.}}$

• $\vec{H}$? : $\frac{\partial}{\partial y}(e^x\cos(xy)) = -xe^x\sin(xy)$

$\ne$

$\frac{\partial}{\partial x}(e^x\sin(xy)) = e^x(\sin(xy) + y\cos(xy))$

$\Rightarrow$ $\vec{H}$ is not conservative.

(Extra page for Question 5.)

(b) Let $f(x,y)$ be s.t. $\nabla f = \vec{G}$.

Then $f_x = 2xe^{-y}$ & $f_y = 2y - x^2e^{-y}$.

$\Rightarrow f(x,y) = \int f_x \, dx = x^2e^{-y} + g(y)$.

But $f_y = 2y - x^2e^{-y} = -x^2e^{-y} + g'(y)$

$\Leftrightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + K$, K in $\mathbb{R}$, say $K = 0$.

Thus $f(x,y) = x^2e^{-y} + y^2$.

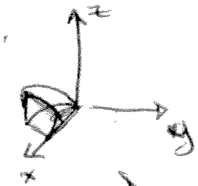
Rk: One could integrate f_y instead & use f_x to find the constant of integration.

(c) By the F.T.L.I., we have

$$\int_C \vec{G}\cdot d\vec{r} = f(1,1) - f(0,0) = \boxed{e^{-1} + 1}$$

1. Consider the solid in the first octant bounded by the planes $z = x$, $z = 0$ and $x = 1$ and the parabolic cylinder $x = y^2$. This solid has a mass density given by the function $\delta(x, y, z) = y$. Find the total mass of this solid.

- In the 1st octant $x, y, z \geq 0$.
- The region of integration is

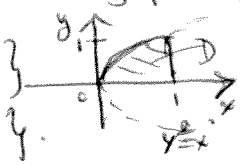


$E = \{(x,y,z) \mid (x,y) \text{ in } D,\ 0 \leq z \leq x\}$,

where D is the projection of E in the xy-plane.

So $D = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$

or $D = \{(x,y) \mid 0 \leq y \leq 1,\ y^2 \leq x \leq 1\}$.



- The total mass is then

$$m = \int_0^1 \int_0^{\sqrt{x}} \int_0^x y\, dz\, dy\, dx \overset{\text{or}}{=} \int_0^1 \int_{y^2}^1 \int_0^x y\, dz\, dx\, dy$$

$$= \boxed{\frac{1}{6}}$$

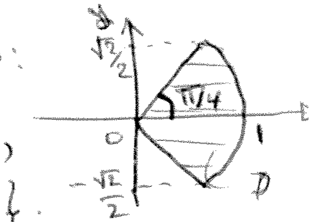
2. Compute the following double integral

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{|y|}^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$$

Hint: Sketch the region of integration in the xy-plane, and then express this region and the integral in a different coordinate system.

- Region of integration:

$D = \{(x,y) \mid -\frac{\sqrt{2}}{2} \leq y \leq \frac{\sqrt{2}}{2},$
$|y| \leq x \leq \sqrt{1-y^2}\}$.



- In polar coords, we have

$D = \{(r,\theta) \mid 0 \leq r \leq 1,\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$,

$x^2 + y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$.

The integral in polar coords is

$$\int_{-\frac{\sqrt{2}}{2}}^{\sqrt{2}/2} \int_{|y|}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy = \int_0^1 \int_{-\pi/4}^{\pi/4} r^2\, r\, d\theta\, dr$$

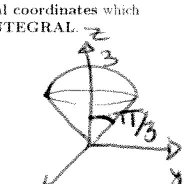
$$= \boxed{\frac{\pi}{8}}$$

3. Consider the solid in the first octant which is bounded by the cone $z = \frac{\sqrt{3}}{3}\sqrt{x^2+y^2}$ and the sphere $x^2 + y^2 + z^2 = 9$. This solid has a mass density given by the function $\delta(x, y, z) = 27 - 4x^2 - 4y^2 - 4z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

- The angle is making the cone with the z-axis satisfies

$\tan(\phi) = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \phi = \frac{\pi}{3}$,

since $0 \leq \phi \leq \pi$.



- So the inclination's $0 \leq \phi \leq \pi/3$.
- In the 1st octant $0 \leq \theta \leq \pi/2$.
- The radius of the sphere is 3, & so $0 \leq \rho \leq 3$.
- In spherical coords $dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$ & $\delta = 27 - 4\rho^2$.
- The total mass is then

$$\boxed{m = \int_0^{\pi/3} \int_0^{\pi/2} \int_0^3 (27 - 4\rho^2)\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \sin(t)\,\vec{\mathbf{i}} + \cos(t)\,\vec{\mathbf{j}} + 4t\,\vec{\mathbf{k}},\quad 0 \leq t \leq \pi/4.$$

- $\vec{r}\,'(t) = \cos(t)\,\vec{i} - \sin(t)\,\vec{j} + 4\,\vec{k}$
- $|\vec{r}\,'(t)| = \sqrt{17}$.
- Arclength:

$$L = \int_0^{\pi/4} |\vec{r}\,'(t)|\, dt = \boxed{\frac{\pi\sqrt{17}}{4}}.$$

5. Among the following three vector fields there is **exactly one** conservative vector field.

$$\vec{F} = (2x + 2xe^{-y})\vec{\imath} - x^2e^{-y}\vec{\jmath}$$
$$\vec{G} = x^2y\vec{\imath} - xy^2\vec{\jmath}$$
$$\vec{H} = e^{xy}\cos(x)\vec{\imath} + e^{xy}\sin(y)\vec{\jmath}$$

(a) Determine which **one** of these vector fields is conservative. If a vector field is not conservative, justify why it is not conservative.

(b) Find a potential function f **for the conservative vector field** in (a).

(c) Evaluate the line integral of **the conservative vector field** in (a) along the curve C, where C is any path forms $(0,0)$ to $(1,1)$.

(a) • $\vec{F}$?: $\frac{\partial}{\partial y}(2x+2xe^{-y}) = -2xe^{-y}$

$$\parallel$$

$$\frac{\partial}{\partial x}(-x^2e^{-y})$$

$\Rightarrow$ $\vec{F}$ is conservative.

• $\vec{G}$?: $\frac{\partial}{\partial y}(x^2y) = x^2 \neq \frac{\partial}{\partial x}(-xy^2) = -y^2$

$\Rightarrow$ $\vec{G}$ is not conservative.

• $\vec{H}$?: $\frac{\partial}{\partial y}(e^{xy}\cos(x)) = xe^{xy}\cos(x)$

$$\nparallel$$

$$\frac{\partial}{\partial x}(e^{xy}\sin(y)) = ye^{xy}\sin(y).$$

$\Rightarrow$ $\vec{H}$ is not conservative.

(b) Let $f(x,y)$ be s.t. $\nabla f = \vec{F}$.

Then $f_x = 2x + 2xe^{-y}$ & $f_y = -x^2e^{-y}$.

$\Rightarrow f(x,y) = \int f_x\,dx = x^2 + x^2e^{-y} + g(y)$.

But $f_y = -x^2e^{-y} \Leftrightarrow -x^2e^{-y} + g'(y) = -x^2e^{-y}$

$\Leftrightarrow g'(y) = 0 \Rightarrow g(y) = K$, K in $\mathbb{R}$,

Take $K = 0$.

Thus $f(x,y) = x^2 + x^2e^{-y}$.

Rk: One could integrate f_y instead & use f_x to find the constant of integration.

(c) By the F.T.L.I., we have

$$\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = \boxed{1 + e^{-1}}$$