University of Ottawa
Department of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 3X - Spring-Summer 2017
Midterm I - V.2
Professor: Abdelkrim El basraoui
Duration: 80 minutes.

Solutions.

Name:______________________________ ID Number:____________________

Instructions: (Please read carefully.)

- This exam has 8 pages and 6 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1, 2, 4, 5 and 6 are worth 7 marks each, and question 3 is worth 5 marks, so organize your time accordingly.
- Answer each question in the space provided, using backs of pages or the extra page at the end if necessary.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
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Signature: ______________________________

Question	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Total
Maximum	7	7	5	7	7	7	40
Score							

1. Find and classify the critical points of the function $f(x,y) = -x^6 - y^6 + 6xy - 1$.

• Critical pts : $\nabla f = \langle -6x^5 + 6y, -6y^5 + 6x \rangle = \langle 0, 0 \rangle$

$$\Leftrightarrow \begin{cases} y = x^5 \\ x = y^5 \end{cases} \Leftrightarrow \begin{cases} x = (x^5)^5 \\ y = x^5 \end{cases} \Leftrightarrow \begin{cases} x(x^{24} - 1) = 0 \\ \& \ y = x^5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \text{ or } x = \pm 1 \\ \& \ y = x^5 . \end{cases}$$

• So we have the critical pts $(0,0)$, $(-1,-1)$ & $(1,1)$.

• Classification : $f_{xx} = -30x^4$; $f_{xy} = f_{yx} = 6$; $f_{yy} = -30y^4$.

Therefore $D(x,y) = 900x^4y^4 - 36$.

* At $(0,0)$: $D(0,0) = -36 < 0 \Rightarrow (0,0)$ is a saddle pt.

* At $(\pm 1, \pm 1)$; $D(\pm 1, \pm 1) = 864 > 0$ & $f_{xx}(\pm 1, \pm 1) = -30 < 0$

$\Rightarrow (-1,-1)$ & $(1,1)$ are local max.

2. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^3 - y^3$ on the unit circle $x^2 + y^2 = 1$.

- Let $g(x,y) = x^2+y^2$. So the constraint is $g(x,y) = 1$.
- Now, $\nabla f = \langle 3x^2, -3y^2 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$

$$\Leftrightarrow \begin{cases} 3x^2 = 2\lambda x \\ -3y^2 = 2\lambda y \end{cases} \Leftrightarrow \begin{cases} x(3x-2\lambda) = 0 \\ y(3y+2\lambda) = 0 \end{cases} \Leftrightarrow \begin{cases} x=0 \text{ or } 3x = 2\lambda \\ y = 0 \text{ or } -3y = 2\lambda \end{cases}$$

- If $x=0$, then the constraint implies that $y^2 = 1 \Leftrightarrow y = \pm 1$ & we have the critical pts $(0, \pm 1)$
- Similarly, if $y=0$ then $x^2 = 1 \Leftrightarrow x = \pm 1$ & we have the critical pts $(\pm 1, 0)$.
- If $3x = 2\lambda = -3y$, then $x = -y$ & from the constaint we get $x^2+x^2 = 2x^2 = 1 \Leftrightarrow x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$. Thus the critical pts $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ & $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.
- Comparison: we have $f(-1,0) = -1 = f(0,1)$; $f(1,0) = 1 = f(0,-1)$; $f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$; $f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$.

Hence f attains its max. value 1 at $(1,0)$ & $(0,-1)$ & its min value -1 at $(-1,0)$ & $(0,1)$.

3. If $f(x,y) = y\,e^{xy}$ and R is the rectangle $[0,2]\times[0,1]$, what is the value of the double integral of f over R, $\iint_R f\,dA$?

4 order simplifies the computation.

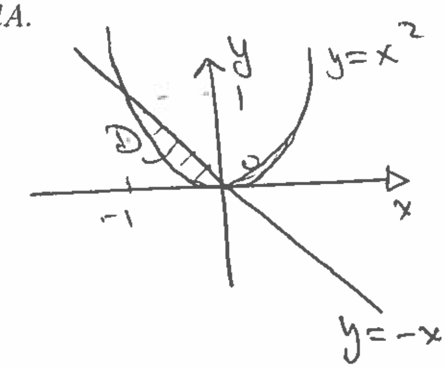
$$\iint_R f\,dA = \int_0^1\int_0^2 y e^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_0^2 dy$$

$$= \int_0^1 (e^{2y}-1)\,dy = \left[\frac{e^{2y}}{2} - y\right]_0^1$$

$$= \boxed{\frac{e^2-3}{2}}$$

(The other way $\int_0^2\int_0^1 y e^{xy}\,dy\,dx$ requires integration by parts.)

4. Let D be the region in the xy-plane bounded by the parabola $y = x^2$ and the line $y = -x$. Sketch the region D then evaluate the following double integral $\iint_D y\,dA$.



(Type I)

• Region: $D = \{(x,y) \mid -1 \leq x \leq 0,\ x^2 \leq y \leq -x\}$

Note that $y=x^2$ & $y=-x$ intersect at $x=0$ & $x=-1$.

• Integration: $$\iint_D y\,dA = \int_{-1}^{0}\int_{x^2}^{-x} y\,dy\,dx$$
$$= \int_{-1}^{0}\left[\frac{y^2}{2}\right]_{x^2}^{-x} dx$$
$$= \int_{-1}^{0}\left(\frac{x^2 - x^4}{2}\right) dx$$
$$= \left[\frac{1}{2}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\right]_{-1}^{0} = \boxed{\frac{1}{15}}$$

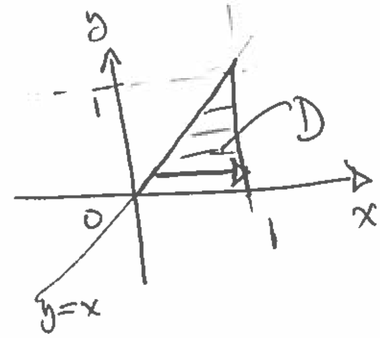
Note that D can be expressed as a type II region too.

$$D = \{(x,y) \mid 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq -y\}.$$

5. Sketch the region of integration then evaluate the following double integral $\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$.

[Hint: use the region to reverse the order of integration.]

- Region: $D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$ (Type II)



As a type I, D is

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$.

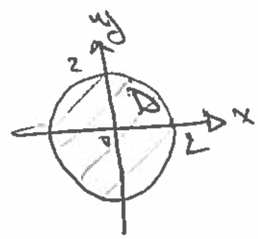
- Integration: reverse the order to get

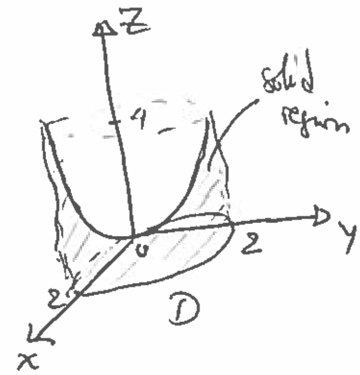
$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx = \int_0^1 \left[y e^{-x^2} \right]_0^x dx$$

$$= \int_0^1 x e^{-x^2}\, dx = \left[-\frac{e^{-x^2}}{2} \right]_0^1 = \boxed{\frac{1-e^{-1}}{2}}$$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 4$, then **evaluate it**.

- Region: $D = \{(x,y) \mid x^2+y^2 \leq 4\}$





- So in polar coords

$$D = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}.$$

- Volume: $V = \iint_D x^2+y^2 \, dA$

$$V = \int_0^2 \int_0^{2\pi} r^2 \; r \, dr \, d\theta \qquad \text{as } x^2+y^2=r^2$$

$$= 2\pi \int_0^2 r^3 \, dr$$

$$= 2\pi \left[\frac{r^4}{4}\right]_0^2 = \boxed{8\pi}$$

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* At $(\pm 1, \pm 1)$: $D(\pm 1, \pm 1) = 864 > 0$ & $f_{xx}(\pm 1, \pm 1) = 30 > 0$

$\Rightarrow$ $(-1,-1)$ & $(1,1)$ are local mins.

2. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^3 + y^3$ on the unit circle $x^2 + y^2 = 1$.

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• Conclusion: We have $f(0,1) = 1 = f(1,0)$; $f(0,-1) = -1 = f(-1,0)$

$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$; $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$.

Hence f attains its max. value 1 at $(1,0)$ & $(0,1)$ & its min. value -1 at $(-1,0)$ & $(0,-1)$.

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$$= \int_0^1 (e^{2x} - 1)\,dx = \left[\frac{e^{2x}}{2} - x\right]_0^1 = \boxed{\frac{e^2 - 3}{2}}$$

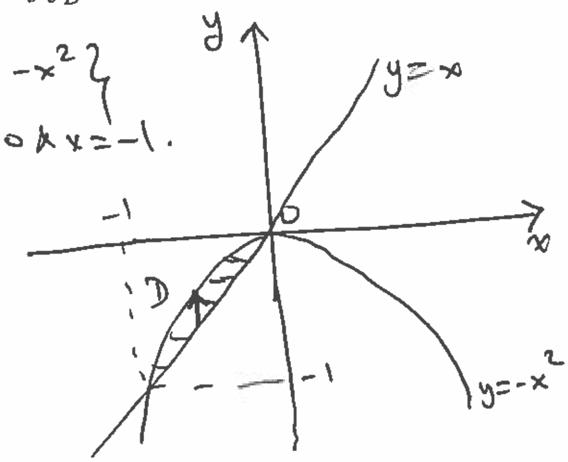
(We would have to use integration by parts if we integrate with respect to x first).

4. Let D be the region in the xy-plane bounded by the parabola $y = -x^2$ and the line $y = x$. Sketch the region D then evaluate the following double integral $\iint_D y\, dA$.

(Type I)

· Region: $D = \{(x,y) \mid -1 \le x \le 0,\ x \le y \le -x^2\}$

· Note that the two curve intersect at $x=0$ & $x=-1$.



· Integration:

$$\iint_D y\, dA = \int_{-1}^{0}\int_{x}^{-x^2} y\, dy\, dx$$

$$= \int_{-1}^{0} \left[\frac{y^2}{2}\right]_{x}^{-x^2} dx = \int_{-1}^{0} \left(\frac{x^4 - x^2}{2}\right) dx$$

$$= \left[\frac{1}{2}\left(\frac{x^5}{5} - \frac{x^3}{3}\right)\right]_{-1}^{0} = \boxed{-\frac{1}{15}}$$

Note that D can expressed as a type II region as well.

$D = \{(x,y) \mid -1 \le y \le 0,\ -\sqrt{-y} \le x \le y\}$.

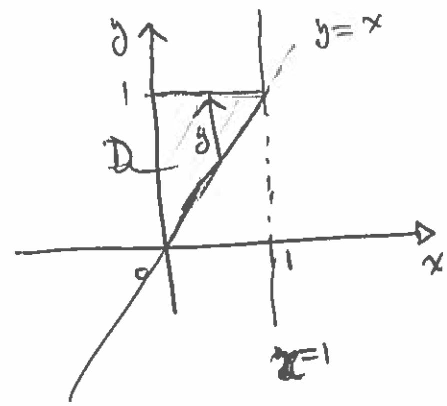
5. Sketch the region of integration then evaluate the following double integral $\int_0^1 \int_x^1 e^{-y^2}\, dy\, dx$.

[Hint: use the region to reverse the order of integration.]

• Region : $D = \{(x,y) \mid 0 \leq x \leq 1 ,\ x \leq y \leq 1\}$ (Type I)

As a type II, D is

$D = \{(x,y) \mid 0 \leq y \leq 1 ,\ 0 \leq x \leq y\}$



• Integration: (order reversing)

$$\int_0^1 \int_x^1 e^{-y^2}\, dy\, dx = \int_0^1 \int_0^y e^{-y^2}\, dx\, dy$$

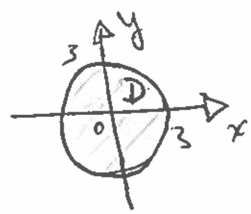
$$= \int_0^1 \left[x e^{-y^2} \right]_0^y dy$$

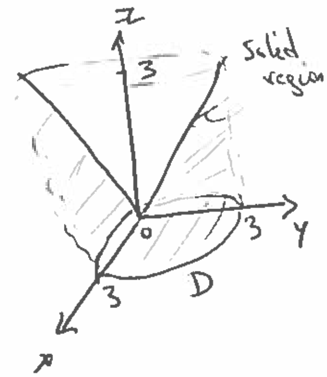
$$= \int_0^1 y e^{-y^2}\, dy$$

$$= \left[-\frac{e^{-y^2}}{2} \right]_0^1$$

$$= \boxed{\frac{1-e^{-1}}{2}}$$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 9$, then **evaluate it**.





• Region: $D = \{(x,y) \mid x^2+y^2 \leq 9\}$

• So in polar coords

$D = \{(r,\theta) \mid 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$.

• Volume: $V = \iint_D \sqrt{x^2+y^2}\, dA$.

$V = \int_0^{2\pi}\int_0^3 r \quad r\,dr\,d\theta$ as $\sqrt{x^2+y^2} = r$.

$= \int_0^{2\pi}\int_0^3 r^2\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^3 d\theta = \boxed{18\pi}$