MAT 1341A F14 Test 1 solns V3

1. An equation for the plane passing through the points $(-1,1,2)$ and $(1,2,3)$, and parallel to the z–axis is:

A. $-3x+7y-2z=3$
B. $2x-z=5$
C. $x-2y=-3$
D. $x-y=1$
E. $2y-z=3$
F. $x+y+z=2$

Such a plane is parallel to the vectors $(1,2,3)-(-1,1,2)=(2,1,1)$ and $(0,0,1)$ and therefore a normal vector would be $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (1,-2,0)$.

The only equation representing a plane with normal parallel to $(1,-2,0)$ in the list above is C. (One easily checks that this plane contains $(-1,1,2)$ & $(1,2,3)$.)

2. Find an equation of the plane which passes through the point $(-7,\ 1,\ 8)$ and which is perpendicular to the line whose (scalar) parametric equations are:

$$x=7-4t,\quad y=2+2t,\quad z=-3+t;\quad t\in\mathbf{R}.$$

A. $2x-4y+z=38$
B. $2x+7y-3z=-71$
C. $2x-4y+z=-28$
D. $-4x+2y+z=38$
E. $-4x+2y+z=-10$
F. $-4x+2y+z=10$

A normal for this plane will be the direction vector of the line above, namely $(-4,2,1)$. One need only note that, as $(-7,1,8)$ belongs to the plane, the only correct equation is D.

3. Parametric equations of the line containing $(1,\ 0,\ -5)$ and which is parallel to the two planes $x - 4y + 2z = 0$ and $-2x - 3y + z = 1$ are:

A. $x = 1 + 2t,\ y = 3t,\ z = 5 + 11t,\ t \in \mathbf{R}$
B. $x = 1 - 10t,\ y = -5t,\ z = -5 + 5t,\ t \in \mathbf{R}$
C. $x = 1 - 2t,\ y = 5t,\ z = -5 + 11t,\ t \in \mathbf{R}$
D. $x = t,\ y = 0,\ z = -5t,\ t \in \mathbf{R}$
E. $x = 1 + 2t,\ y = -3t,\ z = -5 + 11t,\ t \in \mathbf{R}$
F. $x = t,\ y = 0,\ z = 5t,\ t \in \mathbf{R}$

A direction vector d for this line must be perpendicular to both normals above.

Hence $d = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -4 & 2 \\ -2 & -3 & 1 \end{vmatrix} = (2,-5,-11)$. Of course, $(-2,5,11)$ is a direction vector for the line as well. Since it also must contain $(1,0,-5)$, C is correct.

4. One of the following is an equation for the plane with vector parametric description

$$v = (1,1,1) + s(0,0,-1) + t(1,1,0);\ s,t \in \mathbf{R}.$$

Which is it?

A normal vector is $(0,0,-1) \times (1,1,0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (1,-1,0)$.

A. $4x + 36y - 9z = 31$
B. $9x - 2y + 5z = 14$
C. $9x + 18y - 11z = -40$
D. $x - y = 0$
E. $9x + 2y - 2z = 5$
F. $3x - y + 2z = 0$

The only equation representing a plane with a normal $\parallel$ to $(1,-1,0)$ is D, and one can check it contains $(1,1,1)$.

5. Which two of the following are vector parametric descriptions for the plane with equation $x - 2y + z = 4$?

~~I. $v = (0,0,0) + s\ (0,1,2) + t\ (2,1,0); s,t \in \mathbf{R}$.~~ (The plane above does not contain $(0,0,0)$.)

II. $v = (0,-2,0) + s\ (1,0,-1) + t\ (2,1,0); s,t \in \mathbf{R}$.

III. $v = (4,0,0) + s\ (1,1,1) + t\ (1,0,1); s,t \in \mathbf{R}$.

IV. $v = (4,0,0) + s\ (1,0,-1) + t\ (0,1,2); s,t \in \mathbf{R}$.

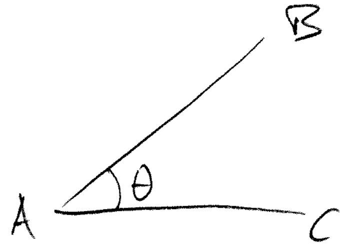
A. I & II
B. I & III
C. I & IV
D. II & III
E. II & IV
F. III & V

When "$v = P + s v_1 + t v_2$" is the vector parametric description of a plane, both vectors v_1 and v_2 must be perpendicular to any normal; in this case $(1,-2,1)$.

Simple checks show that $(1,0,-1)$, $(2,1,0)$, $(1,1,1)$ and $(1,0,-1)$ are $\perp$ to $(1,-2,1)$, but $(1,0,1)$ is not. Hence III cannot be correct, so II & IV are correct.

6. If $A = (1,2,1)$, $B = (2,2,1)$ and $C = (1+\sqrt{3},3,1)$, find the angle $\angle BAC$.

A. $\pi/2$
B. $\pi/3$
C. $\pi/4$
D. $\pi/6$
E. $3\pi/4$
F. $4\pi/3$



$$\cos\theta = \frac{(B-A)\cdot(C-A)}{\|B-A\|\ \|C-A\|} = \frac{(1,0,0)\cdot(\sqrt{3},1,0)}{1\cdot 2} = \frac{\sqrt{3}}{2}.$$ Hence $\theta = \pi/6$

7. If $u = (2,\ 2,\ 2)$ and $v = (1,0,1)$ then $\text{proj}_u\, v =$

$$\frac{v \cdot u}{\|u\|^2}\, u$$

A. $\frac{2\sqrt{3}}{3}(2,2,2)$
B. $\frac{1}{3}(2,2,2)$
C. $(2,0,2)$
D. $\frac{\sqrt{2}}{2}(1,0,1)$
E. $\frac{12}{7}(3,\ 3,\ 3)$
F. $\frac{11}{7}(3,\ 3,\ 3)$

$$= \frac{4}{12} \cdot (2,2,2)$$

$$= \frac{1}{3}(2,2,2)$$

8. Find the volume of the parallelepiped determined by the vectors $u = (1,\ 1,\ -1)$, $v = (2,\ 0,\ 1)$ and $w = (1,\ -1,\ 3)$.

A. -2
B. 4
C. 6
D. 8
E. 16
F. 2

This is $|u \cdot v \times w|$. But

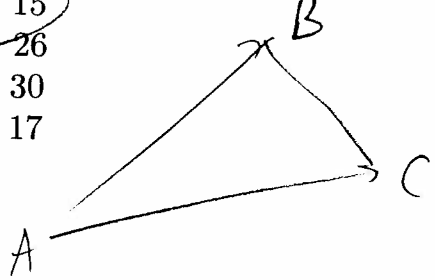
$$v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{vmatrix} = (1, -5, -2).$$

Hence the volume is $|(1,1,-1) \cdot (1,-5,-2)|$

$$= |1 - 5 + 2| = |-2| = 2$$

9. What is the area of the triangle with vertices $\overset{A}{(1,0,1)}$, $\overset{B}{(5,\ 4,\ 3)}$ and $\overset{C}{(3,\ 9,\ 3)}$?

A. 20
~~B. 13~~
C. 15
~~D. 26~~
E. 30
F. 17



The area is $\frac{1}{2}\|(B-A)\times(C-A)\|$

$$= \frac{1}{2}\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 2 \\ 2 & 9 & 2 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\|(-10,-4,28)\|$$

$$= \|(-5,-2,14)\|$$

$$= \sqrt{25+4+196}$$

$$= \sqrt{225}$$

$$= 15$$

$\begin{array}{r} 14 \\ 14 \\ \hline 56 \\ 140 \\ \hline 196 \end{array}$

$225 = 5 \cdot 45$
$= 5 \cdot 5 \cdot 9$
$= 5^2 3^2$
$= 15^2$

10. Let L be the line passing through $\underset{A}{(0,\ 1,\ 1)}$ and $\underset{B}{(1,\ 3,\ 2)}$. The point of intersection of L with the plane $-x+y+z=1$ is:

~~A. (0, 1/2, 1/2)~~
B. (−1/2, 0, 1/2,)
C. (0, 0, 1)
D. (−1/2, 1/2, 0)
E. (0, 1, 0)
F. (−1, 0, −1)

A direction vector for L is $B-A$ $=(1,2,1)$. Hence scalar parametric equations for L are

$$x = 0+t$$
$$y = 1+2t$$
$$z = 1+t.$$

Thus $-x+y+z=1 \iff -t+(1+2t)+1+t=1$
$\iff 2t=-1 \iff t=-\frac{1}{2}$

Hence $(x,y,z)=\left(-\frac{1}{2},0,\frac{1}{2}\right)$.

11. Express the following complex numbers in the form $a + b\,i$:

$(3\neq 0)\ \frac{1}{3} = \frac{\bar{3}}{|3|^2}$

$$z_1 = \frac{i}{1+i} = \frac{i\,(1-i)}{2} = \frac{1}{2} + \frac{1}{2}i$$

$$z_2 = (2+i)(-1+i)$$
$$= -2-1+i$$
$$= -3+i$$

A. $z_1 = 1 + i\ ;\ z_2 = -2 + 2i$

B. $z_1 = -1 - i\ ;\ z_2 = -2 + i$

C. $z_1 = \frac{1}{2} + \frac{1}{2}i\ \ z_2 = -3 + i$

D. $z_1 = -\frac{1}{2} + \frac{1}{2}i\ ;\ z_2 = 3 + i$

E. $z_1 = \frac{1}{4} + 2i\ ;\ z_2 = 1 + 3i$

F. $z_1 = 1 + i\ ;\ z_2 = 2i$

12. Find the polar form of

$$\frac{z_1}{z_2} = \frac{1-\sqrt{3}i}{1+i} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

A. $\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$
B. $\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$
C. $\sqrt{2}(\cos(-7\pi/12) + i\sin(-7\pi/12))$
D. $\sqrt{2}(\cos(5\pi/12) + i\sin(5\pi/12))$
E. $\sqrt{2}(\cos(-\pi/12) + i\sin(-\pi/12))$
F. $\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$

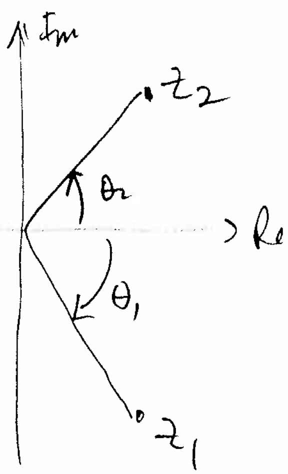
$r_1 = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\cos\theta_1 = \frac{1}{2}\ ;\ \sin\theta_1 = \frac{-\sqrt{3}}{2}\ \therefore\ \theta_1 = -\pi/3$

$r_2 = \sqrt{1^2+1^2} = \sqrt{2}$

$\cos\theta_2 = \frac{1}{\sqrt{2}} = \sin\theta_2 = \frac{\sqrt{2}}{2}$

$\therefore\ \theta_2 = \pi/4$



$\therefore\ \theta_1 - \theta_2 = \pi(-1/3 - 1/4) = -\frac{7\pi}{12}\ ;$

$\frac{r_1}{r_2} = \frac{2}{\sqrt{2}}\ \therefore$ C is correct