Module: Calculus 1

Rates of Change and the Slopes of Curves

Suppose we have a function $y = f(x)$. If we change x from $x = a$ to $x = a + \Delta x = a + h$, for a step or difference in x of $\Delta x = h$, then the value of the function(assuming it's not constant), will change from $f(a)$ to $f(a + h)$.

a graph of a function, showing too points (a,f(a)) and (a+h,f(a+h)) on the curve

If we look at the change in the value of the function, $\Delta f = f(a + h) - f(a)$, relative to the change in the independent variable x, $\Delta x = (a + h) - a = h$, we'll have $\frac{\Delta f}{\Delta x} = \frac{f(a+h) - f(a)}{h}$, which is the average rate of change of the function on the interval $a \leq x \leq a + h$.

The most familiar example of this is probably the velocity of a moving object. If you drive a distance of 120 km in an hour and a half, your average speed for the trip is $\frac{120\text{km}}{1.5\text{h}} = 80\text{km/h}$(and this is a change in position divided by a change in time).

Let's go back to what we had above. Notice that the expression $\frac{\Delta f}{\Delta x} = \frac{f(a+h) - f(a)}{h}$ (called the difference quotient) would represent the slope of a straight line through the two points $(a, f(a))$ and $(a + h, f(a + h))$ on the curve.

A line that passes through two points P and Q on a curve $y = f(x)$ is called a secant line.

a graph of a function with a secant line through point P and Q on the curve

Example

Find the slope of the secant line passing through the points $(-1, 1)$ and $(2, 4)$ on the curve $y = x^2$.

$\frac{\Delta y}{\Delta x} = \frac{4 - 1}{2 - (-1)} = \frac{3}{3} = 1$ and the line is $y = x + 2$

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Can you see that?

Example

Suppose the population of a small town was measured every year from 2001 to 2010.

Year	Population	Year	Population
2001	6210	2006	6975
2002	6347	2007	7087
2003	6523	2008	7214
2004	6704	2009	7326
2005	6851	2010	7472

1. What was the average rate of change of the population over the entire period?
2. How about over 2006 to 2010?

1. $\dfrac{\Delta P}{\Delta t} = \dfrac{7472 - 6210}{2010 - 2001} \approx 140$ people/year
2. $\dfrac{\Delta P}{\Delta t} = \dfrac{7472 - 6975}{2010 - 2006} \approx 124$ people/year

In our driving example above, we had that the average speed over the trip was 80 km/h. It is unlikely that we were actually driving at exactly 80 km/h the entire 1.5 hours -- more likely, at times we were going faster and at other times, slower. Also, the average does not tell us what our speed was at any particular time, which would be an instantaneous velocity (speed at one particular instant).

How could we find the instantaneous rate of change of a function $y = f(x)$ at the value $x = a$? By recognizing that this would correspond to the "slope" of the curve at this point, which would have to be equal to the slope of the tangent line to the curve at the point.

a graph of a function with the tangent line to the curve at the point (a,f(a))

The tangent line to a curve at a point $P = (a, f(a))$ is the straight line that passes through the point and best approximates the curve near the point.

So if we have the graph of the function, we could draw the tangent line and find its slope. (*But this would only be an approximation or estimate as we cannot draw the tangent line with perfect accuracy.*)

If we have a table of values, we could calculate the average rate of change over as short an interval as possible that contains the point of interest (*but again, only an estimate*)

Example

The instantaneous rate of change m of the population of the town in 2004 is approximately

$$m \approx \frac{\Delta P}{\Delta t} = \frac{P(2005) - P(2004)}{2005 - 2004} = \frac{6851 - 6704}{1} = 147 \text{ people/year.}$$

Notice that either way (graphically or numerically), the best we can do is an estimate. Why do we have this problem? Because we only know that the tangent line passes through the point $(a, f(a))$ and nothing else. We cannot calculate the slope (or find the equation) of the line knowing only a single point. So, clearly, if we wish to calculate instantaneous rates of change (*which we do*), we need to figure out a way to do so.
