Module: Calculus 1

Limits

The set of natural numbers is $\mathrm{N} = \{0, 1, 2, 3, \ldots\}$. An infinite sequence is an infinite list of numbers generated by a function $f(n) = a_n$ whose domain is N.

Example

$$2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$$

The general term here is

$a_n = \frac{2}{3^n}$ and as n gets bigger, the values will get smaller and smaller or closer to 0. There is an a_n for all n, no matter how large n is and we can denote the idea of n getting larger and larger without bound by saying that n approaches infinity, which we write as $n \to \infty$. As $n \to \infty$, $a_n \to 0$. We can write this as a limit: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2}{3^n} = 0$.

Example

If $a_n = (-1)^n$, we have the sequence

$$1, -1, 1, -1, \ldots$$

Here, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n$ does not exist because the terms in the sequence are not approaching a single value.

Example

$$1, 2, 4, 8, 16, 32, \ldots$$

Here $a_n = 2^n$ and then $\lim_{n \to \infty} a_n = \lim_{n \to \infty} 2^n = \infty$. This limit does not exist as the terms in the sequence are growing without bound, getting larger and larger, and so they do not approach a single (finite) value.

∞ is not a number -- it represents the idea of unbounded growth.

If $\lim_{n \to \infty} a_n = L$, where L is a unique and finite number, then the sequence $\{a_n\}$ has a limit as $n \to \infty$ and is said to converge to L which means that as n gets larger and larger, the values of a_n approach L.

Given a function $f(x)$, we can also look at what happens as $x \to a$, *ie* take $\lim_{x \to a} f(x)$

a graph of a fucntion

We can approach a from either the left side, where $x < a$, or the right side, where $x > a$. This leads to the limit from the left $\lim_{x \to a^-} f(x)$ and the limit from the right $\lim_{x \to a^+} f(x)$.

Suppose that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but are different (not equal).

a graph of a function with a jump discontinuity at x=a

Then $\lim_{x \to a} f(x)$ cannot exist as $f(x)$ is not approaching a single value as $x \to a$.

What if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$? Then $\lim_{x \to a} f(x)$ must exist and be the same value.

Example

Consider the function $f(x) = x^2 - 2x + 3$. What is $\lim_{x \to 2} f(x)$?

the graph of the function f(x)=x2-2x+3

We can see from the graph that as $x \to 2^-$ (*ie* from the left, $x = 1.9, 1.99, 1.999, \ldots$), the value of the function will increase up to 3. And as $x \to 2^+$ (*ie* from the right, $x = 2.1, 2.01, 2.001, \ldots$), the value of the function decreases to 3.

So we have $\lim_{x \to 2^-} f(x) = 3 = \lim_{x \to 2^+} f(x)$ and so $\lim_{x \to 2} f(x) = 3$.

We could have seen this numerically as well.

x	$f(x)$	x	$f(x)$

1.9	2.81	2.1	3.21
1.99	2.9801	2.01	3.0201
1.999	2.998001	2.001	3.002001

Can you see what these values are saying about $\lim_{x \to 2^-} f(x)$ *and* $\lim_{x \to 2^+} f(x)$ *and hence about* $\lim_{x \to 2} f(x)$*?*

If we look at our example above, we see that $\lim_{x \to 2} f(x)$ is simply the value of $f(x)$ at $x = 2$. We can see that this must be the case because if we traced along the curve from either side of 2 towards 2, we would not experience any breaks in the graph and so we approach $f(2)$. This means that our function $f(x) = x^2 - 2x + 3$ is continuous at $x = 2$.

We say that $f(x)$ is continuous at $x = a$ if three conditions are met.

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function is continuous at $x = a$ if you can draw the graph at $x = a$ without lifting your pencil. A function is called continuous if it is continuous for all x in its domain. If there is a place where there is a break in the graph, we have a discontinuity (and at least one of the three conditions above is violated)