University of Ottawa
MAT 1332. Winter 2017, Midterm Exam 2
Wednesday, March 29^{th}, 2017

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DGD (circle one): 1   2   3   4

First Name ____________________

Family Name ____________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more than others. Make note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed.
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X, and Casio FX-300X) are allowed.
- This exam consists of 7 questions: 3 are multiple choice and 4 are long answer.
  - For the 3 multiple choice questions, only the chosen answer will be marked.
  - For the 4 long answer questions, the correct answer requires justification written legibly and logically: you must convince me that you know your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

  By signing below, you acknowledge that you have ensured that you are complying with the above statement.

  Signature :____________________
- Good Luck!

Student Number: ____________________, Total marks: ______________ out of 40

Problem	1	2	3	4	5	6	7
Marks							

QUESTION 1. (**3 points**) Consider the complex number $z = -1 + i$ and the following statements:

[(i)] $|z| = \sqrt{2}$.
(ii) $z = |z|e^{((\pi i)/4)}$.
(iii) $z = |z| \left(\cos\left(\frac{(\pi)}{4}\right) + i \sin\left(\frac{(\pi)}{4}\right)\right)$.
[(iv)] $\bar{z} = -1 - i$.
[(v)] $\frac{1}{z} = -\frac{1}{2} - \frac{i}{2}$.

Which of the following assertions is correct:

A: Only (i) is true. B: (i) and (iii) are true. C: (i), (ii), and (iv) are true.

[D: (i), (iv), and (v) are true.] E: (ii) and (iii) are true.

The answer is D.

Since z would be in quadrant II in the cartesian plane the angle it forms with the positive x-axis is $3\pi/4$. Thus (ii) and (iii) are incorrect.

Question 2. (**3 points**) Consider the following matrices:

$$X = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad , \quad Y = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{et} \quad Z = \begin{bmatrix} 1 & 5 \\ -1 & 0 \\ 3 & 2 \end{bmatrix} .$$

and the following statements:

(i) XY is well defined.
(ii) XZ is well defined.
(iii) $YZ^\top$ is well defined.
(iv) $ZX^\top$ is well defined.
(v) The matrix Y is not invertible.
(vi) $Y + XZ$ is well defined.

Which of the following assertions is correct:

A: (i), (iii), and (vi) are true; B: (iii), (iv), and (v) are true; C : (ii), (iii), and (v) are true;

D: (ii) and (iv) are true; E: (ii) and (vi) are true.

The answer is C.

- (i) is incorrect. XY is undefined since the number of columns of X (=3) does not match the number of rows of Y (=2).
- (iv) is incorrect. ZX^T is undefined since the number of columns of Z (=2) does not match the number of rows of X^T (=3).
- (vi) is incorrect. Y which is 2×2 cannot be added to XZ which is 3×2.

QUESTION 3. (**3 points**) Which of the following values is an eigenvalue for the matrix. Circle every answer that applies.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A: 0; B: 1; $\boxed{\text{C: 2;}}$ D: 3; E: 4.

The answer is C.

$$\begin{aligned} & \det(A - \lambda I) \\ = \; & (2-\lambda)(2-\lambda)(2-\lambda) + 0 + 0 - 0 - (2-\lambda)(-1)(-1) - (-1)(-1)(2-\lambda) \\ = \; & (2-\lambda)((2-\lambda)(2-\lambda) - 1 - 1) \\ = \; & (2-\lambda)(\lambda^2 - 4\lambda + 2) \end{aligned}$$

Thus we have that $\lambda = 2$ or $\lambda = 2 \pm \sqrt{3}$. are the eigenvalues.

Question 4. (**6 points**)

(a) (**1 point**) Consider the system

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 3x_3 & = & 2 \\ 2x_1 & & & - & x_3 & = & 2 \\ 3x_1 & - & 8x_2 & + & x_3 & = & 0 \end{array}$$

Determine the augmented matrix associated to this system.

$$\begin{bmatrix} 1 & -1 & 3 & | & 2 \\ 2 & 0 & -1 & | & 2 \\ 3 & -8 & 1 & | & 0 \end{bmatrix}$$

Marking Scheme: -0.25 per error in the matrix

(b) (**5 points**) Each of the following matrices represents the row echelon form (REF) of the augmented matrix of a linear system of equation. In each case determine the number of solutions the system admits and if the system is consistent determine its solution set.

(i) $\begin{bmatrix} 1 & 2 & 4 & | & 8 \\ 0 & 0 & 3 & | & 9 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 2 & 4 & | & 8 \\ 0 & 0 & \boxed{3} & | & 9 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since, there is no leading coefficient in the last column, this system is consistent. Moreover, since there is no leading coefficient in the second variable column (call it x_2), x_2 is free and the system would admit an infinite number of solutions. Since

$$\begin{bmatrix} 1 & 2 & 4 & | & 8 \\ 0 & 0 & 3 & | & 9 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 2 & 0 & | & -4 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since x_2 is free, we set $x_2 = t$, $t \in \mathbb{R}$. Also we can deduce that

$$x_1 = -4 - 2t$$

$$x_3 = 3$$

Therefore the solution set:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 - t \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; t \in \mathbb{R}$$

Marking Scheme: 0.5 mark for correct number of solutions, 0.5 mark for correct justification, 0.5 mark for a correct method for solving (either RREF or backsubstitution), 0.5 for solution set which follows from their technique

(ii) $\begin{bmatrix} 1 & 2 & 4 & | & 8 \\ 0 & 3 & 1 & | & 9 \\ 0 & 0 & 4 & | & -12 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 2 & 4 & | & 8 \\ 0 & \boxed{3} & 1 & | & 9 \\ 0 & 0 & \boxed{4} & | & -12 \end{bmatrix}$$

Since, there is no leading coefficient in the last column, this system is consistent. Moreover, since there is a leading coefficient in every variable column the system would admit a unique solution.

Since $\begin{bmatrix} 1 & 2 & 4 & | & 8 \\ 0 & 3 & 1 & | & 9 \\ 0 & 0 & 4 & | & -12 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 0 & 0 & | & 12 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$

Thus, we can deduce that

$$x_1 = 12$$

$$x_2 = 4$$

$$x_3 = 3$$

Therefore the solution set:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ -3 \end{bmatrix}$$

Marking Scheme: 0.5 mark for correct number of solutions, 0.5 mark for correct justification, 0.5 mark for a correct method for solving (either RREF or backsubstitution), 0.5 for solution set which follows from their technique

(iii) $\begin{bmatrix} 1 & 2 & 4 & | & 8 \\ 0 & 3 & 1 & | & 9 \\ 0 & 0 & 0 & | & -4 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 2 & 4 & | & 8 \\ 0 & \boxed{3} & 1 & | & 9 \\ 0 & 0 & 0 & | & \boxed{\text{-4}} \end{bmatrix}$$

Since, there is a leading coefficient in the last column, this system is inconsistent. Therefore, the system does not admit any solutions.

Marking Scheme: 0.5 mark for correct number of solutions, 0.5 mark for correct justification.

QUESTION 5. (**5 points**)

a) (**3 points**) The following matrix is invertible. Determine its inverse.

$$B = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 := R_2 + (-2)R_1 \\ R_3 := R_3 + (-1)R_1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & 5 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 := R_3 + (2)R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 := (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 := R_1 + (-3)R_3 \\ R_2 := R_2 + (3)R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 6 & -14 & 3 \\ 0 & 1 & 0 & -5 & 13 & -3 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 := R_1 + (-2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 16 & -40 & 9 \\ 0 & 1 & 0 & -5 & 13 & -3 \\ 0 & 0 & 1 & -2 & 5 & -1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 16 & -40 & 9 \\ -5 & 13 & -3 \\ -2 & 5 & -1 \end{bmatrix}$$

Marking Scheme: 0.5 mark for super augmented matrix, 1 mark for correct row operations being used, 0.5 mark arithmetic, 1 mark for a correct inverse matrix (0.5 if the inverse is incorrect but they have checked their worked and are aware of this fact, 0 marks if an incorrect inverse is found without any justification)

b) (**2 points**) Compute the determinant of the given matrix and determine if it is invertible. Justify your answer.

$$A = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (3)(2)(0) + (0)(-1)(1) + (3)(-1)(1) - (3)(2)(1) - (3)(-1)(1) - (0)(-1)(0) \\ &= 0 + 0 - 3 - 6 + 3 - 0 \\ &= -6 \end{aligned}$$

Since $\det(A) \neq 0$, A is invertible.

Marking Scheme: 1 mark correct determinant (0.5 only if there is an arithmetic mistake but correct formula was used) , 1 mark for conclusion which follows from their determinants result.

Question 6. (**11 points**) Consider the following linear system of differential equations

$$\begin{aligned} x' &= 7(-x+y) \\ y' &= x-y. \end{aligned}$$

(a) (**6 points**) Compute the eigenvalues and their associated families of eigenvectors for the matrix of coefficients associated to this system.

The matrix of coefficients is

$$A = \begin{bmatrix} -7 & 7 \\ 1 & -1 \end{bmatrix}$$

Thus we solve for λ such that

$$\begin{aligned} \det(A-\lambda I) &= 0 \\ \det\left(\begin{bmatrix} -7-\lambda & 7 \\ 1 & -1-\lambda \end{bmatrix}\right) &= 0 \\ (-7-\lambda)(-1-\lambda)-(7)(1) &= 0 \\ \lambda^2+8\lambda &= 0 \\ \lambda(\lambda+8) &= 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -8$

- For $\lambda_1 = 0$, we find the associated eigenvectors:

$$\begin{aligned} \begin{bmatrix} A-(0)I & \mid & 0 \end{bmatrix} &= \begin{bmatrix} -7 & 7 & \mid & 0 \\ 1 & -1 & \mid & 0 \end{bmatrix} \\ &\to \dots \to \begin{bmatrix} 1 & -1 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix} \end{aligned}$$

  This results in the family of eigenvectors given by:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}; t \in \mathbb{R}, t \neq 0$$

- For $\lambda_2 = -8$, we find the associated eigenvectors:

$$\begin{aligned} \begin{bmatrix} A-(-7)I & \mid & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 7 & \mid & 0 \\ 1 & 7 & \mid & 0 \end{bmatrix} \\ &\to \dots \to \begin{bmatrix} 1 & 7 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix} \end{aligned}$$

  This results in the family of eigenvectors given by:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -7t \\ t \end{bmatrix} = t\begin{bmatrix} -7 \\ 1 \end{bmatrix}; t \in \mathbb{R}, t \neq 0$$

Marking Scheme: 2 marks for finding eigenvalues (0.5 for the set-up det = 0, 0.5 for det expression , 1 mark solving for the eigenvalues), 2 marks for each eigenvector (0.5 for correct system set up, 0.5 for solving technique, 1 mark for the family of eigenvectors (-0.5 if they forgot to exclude trivial solution))

(b) (**2 points**) Determine the general solution of this linear system of differential equations.

$$\begin{aligned} X(t) &= C_1 e^{(0)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} -7 \\ 1 \end{bmatrix} \\ X(t) &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} -7 \\ 1 \end{bmatrix} \end{aligned}$$

Marking Scheme: 1 mark for correct general solution format, 1 mark for inserting correct eigenvalues and eigenvectors in it

(c) (**3 points**) Determine the particular solution associated to the initial condition $x(0) = 9$ and $y(0) = 1$.

$$\begin{aligned} X(t) &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} -7 \\ 1 \end{bmatrix} \\ X(0) &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-8(0)} \begin{bmatrix} -7 \\ 1 \end{bmatrix} \\ [9//1] &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -7 \\ 1 \end{bmatrix} \end{aligned}$$

To find C_1 and C_2 we solve

$$\begin{bmatrix} 1 & -7 & | & 9 \\ 1 & 1 & | & 1 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -1 \end{bmatrix}$$

which implies $C_1 = 2$ and $C_2 = -1$.

Thus the particular solution is:

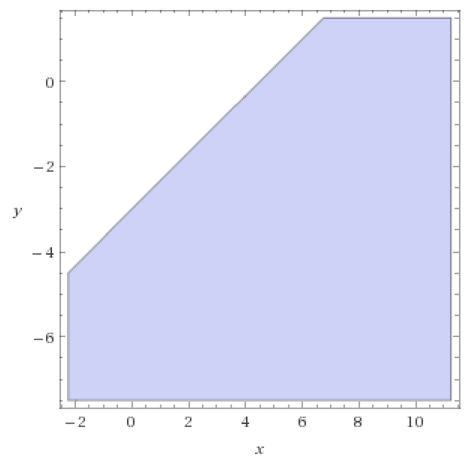
$$X(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - e^{-8t} \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

Marking Scheme: 1 mark for correct use of initial condition to obtain determinant, 1 mark for solving for C_1 and C_2, 1 mark for presenting the particular solution.

QUESTION 7. (**9 points**) Consider the two-variable function $f(x, y) = \sqrt{2x - 3y - 9}$.

(a) (**2 points**) Determine the domain of definition of f. Display the domain with a graph.

We need that $2x - 3y - 9 \geq 0 \Rightarrow y \leq \frac{2}{3}x - 3$. This is represented in the following graph:



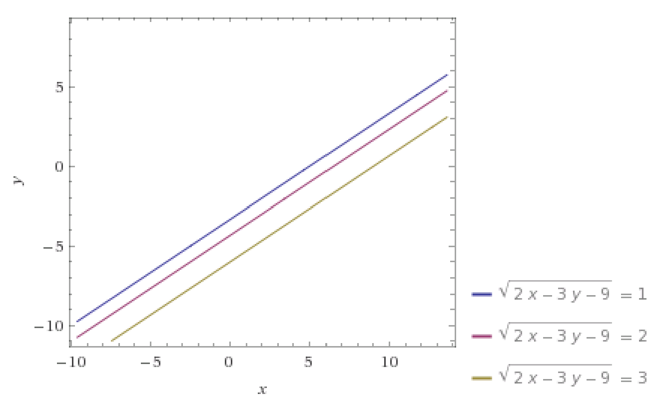
Marking Scheme: 1 mark correct conditions on domain identified , 1 mark for correct graphical representation of what they proposed as a domain.

(b) (**1 point**) Determine the range of f,

Since $2x - 3y - 9 \geq 0$, and the image of a standard square root function is $[0, \infty[$ over a domain of $[0, \infty[$, we have that the range of $f(x, y)$ is $[0, \infty[$.

Marking Scheme: 1 mark for correct range

(c) (**2 points**) Sketch three distinct level curves for this function.



Marking Scheme: 1 mark correct for showing what the equation of a particular level curve must be, 1 mark for correct graphical representation of 3 distinct curves.

(d) (**4 points**) Find the equation of the tangent plane to the surface of $f(x, y)$ at the point $(2, -3, 2)$.

We compute

$$f_x(x, y) = \frac{1}{2}(2x - 3y - 9)^{-1/2}(2) = \frac{1}{\sqrt{2x - 3y - 9}}$$

$$f_y(x, y) = \frac{1}{2}(2x - 3y - 9)^{-1/2}(-3) = \frac{-3}{2\sqrt{2x - 3y - 9}}$$

We know that

$$\begin{aligned} z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z - (2) &= \frac{1}{2}(x - 2) + \frac{-3}{2(2)}(y - (-3)) \\ z - 2 &= \frac{1}{2}x - 1 - \frac{3}{4}y - \frac{9}{4} \\ -2 + 1 + \frac{9}{4} &= \frac{1}{2}x - \frac{3}{4}y - z \\ \frac{5}{4} &= \frac{1}{2}x - \frac{3}{4}y - z \end{aligned}$$

Marking Scheme: 1 mark per correct partial derivatives, 1 mark for the correct format of a tangent plan, 1 mark for subbing in the correct information into the tangent plane formula.