Module: Calculus 1

Limits and Continuity

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is any constant, we have the following limit properties.

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} (cf(x)) = c\left(\lim_{x \to a} f(x)\right)$
5. $\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$, if n is rational
8. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, provided the root on the right hand side exists

These allow us to find many limits (which are really then being calculated by substitution).

Example

1. $\lim_{x \to -3} 27 = 27$
2. $\lim_{x \to -3} x = -3$
3. $\lim_{x \to 2} (2x^3 + 3x - 4) = 2\left(\lim_{x \to 2} x\right)^3 + 3\left(\lim_{x \to 2} x\right) - \lim_{x \to 2} 4 = 2(2)^3 + 3(2) - 4 = 18$
4. $\lim_{x \to 1} \dfrac{5x}{x-1} = \dfrac{5\left(\lim_{x \to 1} x\right)}{\lim_{x \to 1}(x-1)} = \dfrac{5}{0}$ does not exist
5. $\lim_{x \to 4} \sqrt{\dfrac{x^2+1}{x+2}} = \sqrt{\dfrac{\left(\lim_{x \to 4} x\right)^2 + 1}{\left(\lim_{x \to 4} x\right) + 2}} = \sqrt{\dfrac{4^2+1}{4+2}} = \sqrt{\dfrac{17}{6}}$

What we can recognize is that algebraic functions, rational functions and polynomials are all continuous everywhere on their domains and hence limits (within the domains) can be calculated by simply substituting.

Example

1. $\lim_{x \to -1} (2x^2 + 7x - 2) = 2(-1)^2 + 7(-1) - 2 = -7$
2. $\lim_{x \to 3} \frac{x+7}{x-2} = \frac{3+7}{3-2} = \frac{10}{1} = 10$
3. $\lim_{x \to 2} \sqrt{x-7} = \sqrt{-5}$ does not exist (*2 is not in the domain*)
4. $\lim_{x \to 2} \sqrt{2x+1} = \sqrt{2(2)+1} = \sqrt{5}$

We said that $f(x)$ is continuous at $x = a$ if $f(a)$ is defined, $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$ and we said that this would correspond to being able to draw the graph of $f(x)$ without lifting our pencil or without there being a break in the graph at $x = a$. But what if there is a discontinuity at $x = a$? Consider the following graphs.

a graph of a function that is continuous at x=a but the definition of the function changes at x=a

$f(a)$ defined, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$ and thus $f(x)$ is continuous at $x = a$.

a graph of a function with a jump discontinuity at x=a

$f(a)$ is defined, $\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ does not exist and thus $f(x)$ is discontinuous at $x = a$. This is called a jump discontinuity.

another function with a jump discontinuity at x=a, with f(a) defined at an isolated point

$f(a)$ is defined, $\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ does not exist and thus $f(x)$ is discontinuous at $x = a$. This is called a jump discontinuity.

a graph of a function with a hole at x=a and f(a) is not defined

$f(a)$ is not defined, but $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists, but $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.

another function with a hole discontinuity at x=a, with f(a) defined at an isolated point

$f(a)$ is defined, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$, so $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.

a graph of a function with a vertical asymptote at x=a

$f(a)$ is not defined, $\lim_{x \to a^-} f(x)$ does not exist, $\lim_{x \to a^+} f(x)$ does not exist and $\lim_{x \to a} f(x)$ does not exist. $f(x)$ is discontinuous at $x = a$. This is called a vertical asymptote.

Example

Consider the function $f(x) = \begin{cases} 1 + x & x < 2 \\ 2 & x = 2 \\ x^2 & x > 2 \end{cases}$.

This is an example of a piecewise defined function.

the graph of the function in the exemple

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 + x) = 3$,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$,

so $\lim_{x \to 2} f(x)$ does not exist (so $f(x)$ cannot be continuous at $x = 2$), while $f(2) = 2$,

but $f(x)$ has a jump discontinuity at $x = 2$.

Consider the following limits.

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$
2. $\lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1}$
3. $\lim_{x \to 2} \frac{(x - 1)^2 - 1}{x - 2}$

Substituting into all of these will yield the indeterminate form $\frac{0}{0}$ (which is undefined). We can evaluate limits like these by doing certain manipulations.

1. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

$= \lim_{x \to 3} \frac{(x + 3)(x - 3)}{x - 3}$ *(factor)*

$= \lim_{x \to 3} (x + 3)$*(cancel common factor -- allowed because x ≠ 3)*

$= 6$

2. $\lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1}$

$= \lim_{x \to 1} \frac{\sqrt{x + 3} - 2}{x - 1} \times \frac{\sqrt{x + 3} + 2}{\sqrt{x + 3} + 2}$ *(rationalize the numerator)}*

$= \lim_{x \to 1} \frac{(x + 3) - 4}{(x - 1)(\sqrt{x + 3} + 2)}$

$= \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x + 3} + 2)}$

$= \lim_{x \to 1} \frac{1}{\sqrt{x + 3} + 2}$ *(cancel common factor)*

$= 1/4$

3. $\lim_{x \to 2} \frac{(x-1)^2 - 1}{x-2}$

$= \lim_{x \to 2} \frac{x^2 - 2x + 1 - 1}{x - 2}$ *(expand)*

$= \lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$

$= \lim_{x \to 2} \frac{x(x-2)}{x-2}$ *(factor)*

$= \lim_{x \to 2} x$ *(cancel common factor)*

$= 2$

(All of these discontinuities are removable.)

Example

Is $f(x) = \frac{x^2 - 2x - 3}{x^2 + 5x + 4}$ continuous at $x = -1$? Does the limit exist at $x = -1$?

$f(-1) = \frac{0}{0}$, so $f(x)$ is not defined at $x = -1$ and so it cannot be continuous there.

$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x^2 + 5x + 4}$

$\lim_{x \to -1} \frac{(x-3)(x+1)}{(x+1)(x+4)}$

$\lim_{x \to -1} \frac{x-3}{x+4}$

$= -4/3$

so yes, $\lim_{x \to -1} f(x)$ exists.

And this discontinuity is removable -- so the graph of the function would have a hole at point $(-1, -4/3)$.

Videos

Introduction to Limits

Limit Examples (part 1)

Limit Examples (part 2)

Limit Examples (part 3)

Limit Examples (part 4)

More Limits