MAT 1330: Calculus for Life Sciences

A course based on the book Modeling the dynamics of life by F.R. Adler

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August 3, 2016

Preliminaries and expectations

- To be good at math (in fact at anything): practice, and know yourself!
- Find your resources.
- Do 10 problems every day.
- Check with your friends, your TA, the math help centre, and with me.
- I am here to facilitate your learning.
- If you think you understand everything in class, think again.
- Learning math is learning a language, an powerful one. It shows over time. You need to build vocabulary and learn the grammar.
- School is more 'how' and less'why'; University is more 'why' and less 'how'. It is your responsibility to practice the 'how' until you know.
- Draw a picture whenever you can!
- This set of notes provides a short summary for each class with reference to the corresponding book chapter and practice problems. Use it well.

1 Review of the basics

GOAL: Recall and practice all the high-school material that you will need for this course.

SKILLS:

Algebraic manipulations I: powers, roots, exponentials, logarithms

- simplify $\frac{\sqrt{x^{1/2}y^5}}{x^5y^{1/4}}$, where $x, y, z > 0$
- solve $2^{x+3} = 16^{2x-1}$ and $2^{2x+3} = 3^{4x-1}$
- solve $\log(x+5) - \log(x-1) = \log(x+1)$ and $2\ln(x) - \ln(x+4) = \ln(2)$

Algebraic manipulations II: simplify multiple fractions, rationalize a denominator

- simplify $\frac{4+\frac{1}{k}}{\frac{5}{k}-2}$
- rationalize the denominator $\frac{1}{\sqrt{10}-3}$

Polynomials: solve quadratic equations, factor a polynomial, long division

- solve $\frac{1}{x} + \frac{1}{x^2} = 1$
- solve $m = \sqrt{m+6}$
- solve $\frac{4x}{1+x} = 3x$
- factor $x^3 + 1000$
- divide $x^3 + x^2 + \frac{5}{4}x + 3$ by $x + \frac{3}{2}$

Inequalities: handle and solve inequalities, absolute values

- easy: $\frac{x}{2} - 3 > 5$
- harder $\frac{2}{x} - 3 > 5$
- solve $|x^2 - 5| = 1$
- solve $|\frac{x}{2} - 3| > 5$

Check out the history and applications of the quadratic equation on
`https://plus.maths.org/content/os/issue29/features/quadratic/index`

1.1 Practice makes progress

Question 1: Simplify the following expressions.

$$\text{(a)}\ \frac{\left(x^{1/2}y^{1/3}\right)^{-1/2}}{x^2y^3} \qquad \text{(b)}\ \frac{\left(x^{1/2}y^{1/3}\right)^{-1/2}}{x^3y^2} \qquad \text{(c)}\ \left(\frac{x^{3/4}y^3}{x^{-1/4}y}\right)^2\left(\frac{xy}{\sqrt[3]{y}}\right)$$

Question 2: Rationalize the denominator.

$$\text{(a)}\ \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}+\sqrt{5}} \qquad \text{(b)}\ \frac{\sqrt{4}+\sqrt{x}}{\sqrt{4}-\sqrt{x}} \qquad \text{(c)}\ \frac{\sqrt{3}-2}{\sqrt{3}+2}$$

Question 3: Simplify the following expressions.

$$\text{(a)}\ \frac{1-x^3}{1-x} \qquad \text{(b)}\ \frac{1+x^3}{1+x} \qquad \text{(c)}\ \frac{8-x^3}{x-2}$$

Question 4: Simplify the following expressions.

$$\text{(a)}\ \frac{1}{\frac{1}{x}+\frac{1}{x-1}} \qquad \text{(b)}\ \frac{a^{-1}+x^{-2}}{a^{-2}-x^{-1}} \qquad \text{(c)}\ \frac{1}{\frac{3}{x}-\frac{1}{x+3}}$$

Question 5: Find all solutions of the following equations.

$$\text{(a)}\ \log(x+3)+\log(x+4)=\log(6) \qquad \text{(b)}\ \log(x+3)+\log(x+5)=\log(3)$$
$$\text{(c)}\ \log(x+3)+\log(x+4)=\log(2) \qquad \text{(d)}\ \log(x+2)+\log(x+3)=\log(2)$$

Question 6: Find all values of k for which the equation has only one solution.

$$\text{(a)}\ x^2+2kx+9k-8=0 \qquad \text{(b)}\ x^2+2kx+3k+40=0 \qquad \text{(c)}\ x^2+2kx+11k-28=0$$

Question 7: Find all solutions of the following equations.

$$\text{(a)}\ 3x+\frac{4x}{x+1}=5x \qquad \text{(b)}\ 3x+\frac{5x}{x+2}=5x \qquad \text{(c)}\ 3x+\frac{7x}{x+2}=5x$$

Question 8: Find all x for which the following inequality is true.

$$\text{(a)}\ \frac{1}{x-3}>\frac{1}{x+2} \qquad \text{(b)}\ \frac{1}{x-4}<\frac{1}{x+3} \qquad \text{(c)}\ \frac{1}{x-5}<\frac{1}{x+1}$$

Question 9: Solve for x in terms of y and z.

$$\text{(a)}\ \frac{1}{x}+\frac{2}{y}=\frac{7}{z} \qquad \text{(b)}\ \frac{1}{y}-\frac{2}{x}=\frac{5}{z} \qquad \text{(c)}\ \frac{1}{y}-\frac{2}{x}=\frac{5}{z}$$

Question 10: Find all solutions of the following equation.

$$\text{(a)}\ |x^3-5|=5 \qquad \text{(b)}\ |x^3-4|=4 \qquad \text{(c)}\ |2-x^3|=2$$

Question 11: Solve the following equations.

1. Find the value of x so that $\log(x^4y^3) = 18$ when $\log(y) = 2$.
2. Find the value of x so that $\log(x^3y^5) = 9$ when $\log(y) = 3$.
3. Find the value of x so that $\log(x^3y^5) = 21$ when $\log(y) = 3$.

Question 12: Which of the following is **not true** for positive real numbers a, b?
A: $\log(a^xb^y) = x\log a + y\log b$; B: $\left(\frac{a}{b}\right)^x = a^xb^{-x}$; C: $a^b = 10^{b\log a}$;
D: $\log(a^x + b^x) = x\log(a+b)$; E: $\sqrt{a^x} = a^{x/2}$

2 Review of functions

GOAL: Recall and practice all the high-school material that you will need for this course.

- Notation: $y = f(x)$, domain, range, assignment, name
- Polynomial functions
  - linear: $y = f(x) = mx + b$
  - quadratic: $y = f(x) = ax^2 + bx + c$ (find zeros!)
  - higher order: find zeros, long division
- Potential characteristics of functions
  - symmetry: even $f(x) = f(-x)$ or odd $f(x) = -f(-x)$ functions
  - transformations: $y = f(ax + b) + c$
  - composition of functions: $(f \circ g)(x) = f(g(x))$
- Rational functions: fractions of polynomials
  - domain of definition, e.g. $y = f(x) = \frac{1}{x+1}$
- Functions and inequalities
  - comparing graphs: is $f(x) > g(x)$?, e.g. $f(x) = 3x - 2$ and $g(x) = x^2$
  - root functions and domains, e.g. $y = f(x) = \sqrt{\frac{2}{x} - 8}$
  - rational functions, e.g. $\frac{4-x}{2x+1} > 5$
- Absolute value
  - definition: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.
  - for the graph: flip the part of the graph that is below the x-axis above it
- Trigonometric functions
  - $\sin(x)$, $\cos(x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$
  - $\sin^2(x) + \cos^2(x) = 1$
- Exponential and logarithm functions
  - $y = f(x) = e^x = \exp(x)$, $y = e^{-x} = \frac{1}{e^x}$, range?
  - inverse function: $y = \ln(x)$ if $e^y = x$, domain of definition?

2.1 Practice makes progress

A good portion of this material is covered in the textbook for this course, namely in sections 1.3 and 1.4 (second edition) [0.2 and 0.3 (first edition)]. Several nice and instructive examples for how to use functions in simple models are presented in sections 2.1, 2.2, and 2.3 (second edition) [1.1, 1.2, and 1.3 (first edition)]. Selected exercises from the book are:

- second edition: 1.3: 3-8, 13-27; 1.4: 13-20, 33-38, 50-59
- first edition: 0.2: 3-8, 13-27; 0.3: 13-20, 33-38, 50-59

Question 1: Find $f(a+3)$ when the function f is given by

$$\text{(a)}\ \ f(x) = \frac{x^2+1}{x-1}, \qquad \text{(b)}\ \ f(x) = \frac{x^2-2}{x+1}, \qquad \text{(c)}\ \ f(x) = \frac{x^2-4}{x+1}.$$

Question 2: Which of the following functions are even functions?

$$\text{(i) } x^4+3x^2-1; \qquad \text{(ii) } 4x^3+2x-6; \qquad \text{(iii) } |x|+x^2;$$
$$\text{(iv) } e^{-x}; \qquad \text{(v) } x^4+(x+2)^2-4$$

Question 3: Which of the following statements is true?
(i) The graph of $f(x+2)$ is obtained by shifting the graph of $f(x)$ to the left by two units.
(ii) The graph of an odd function is symmetric with respect to the y-axis.
(iii) The graph of every quadratic function crosses the x axis twice.
(iv) The graph of every function crosses the y-axis exactly once.
(v) An increasing function can cross the x-axis at most once.

Question 4: Find the domain of definition of each of the following functions.

$$\text{(a)}\ \ f(x) = \frac{x+1}{x^2-2} \qquad \text{(b)}\ \ f(x) = \frac{x+2}{x(x^2-3)} \qquad \text{(c)}\ \ f(x) = \frac{x+3}{x^2-5}$$

Question 5: Find the value of the composition.

1. If $f(x) = 5x-1$ and $g(x) = -3x+2$ what is $(f \circ g)(1)$?
2. If $f(x) = 5x-1$ and $g(x) = 3x-4$ what is $(f \circ g)(1)$?
3. If $f(x) = 7x-4$ and $g(x) = -3x+1$ what is $(g \circ f)(1)$?

Question 6: Suppose we have a function $y = f(x)$ and we want to shift its graph up by 4 units and to the left by six units. How do we have to choose a and b in $y = f(x+a)+b$ to accomplish that?

Now go and try your re-discovered pre-calculus skills online. There are various test sites where you can get even more practice problems for free. (If some questions focus on topics very different from those covered here, you may skip them as they will not be central to this course.)

The UOttawa diagnostic tool
http://science.uottawa.ca/mathstat/en/why-study/student-resources/diagnostic-test

Some US schools

http://mdtp.ucsd.edu/test_new/index.php

https://math.berkeley.edu/courses/choosing/placement-exam

http://moltest.missouri.edu/mucgi-bin/munew.cgi?variable=calcready

If you find more resources online, please let me know and I will include them in this and/or future courses.

3 Discrete-time dynamical systems (DTDS) - part I

GOAL: Find a mathematical description for dynamic processes in nature.

IDEA: Project forward from one generation to the next. A function describes how the initial measurement and the final measurement in an experiment are related. Then iterate.

Definition: A *DTDS* is an assignment of the form $x_{t+1} = f(x_t)$, where x_t is the state of the system at time t and f is the updating function. An *initial condition* is a value x_0. Sometimes a DTDS is called recursion.

Examples:

- Constant addition between generations (e.g. trees): $x_{t+1} = x_t + c$
- Constant multiple between generations (e.g. bacteria): $x_{t+1} = rx_t$
- Constant decay and addition (e.g. medication): $x_{t+1} = rx_t + c$ with $0 < r < 1$

Definition: A *solution* of a DTDS is a sequence of numbers that describes the future of the system, i.e. $\{x_0, x_1, x_2, \dots\}$ with the property that $x_{t+1} = f(x_t)$.

Note: A solution is not a single number but an entire infinite sequence. A solution depends on the initial condition.

General Result: The general solution of a linear DTDS

$$x_{t+1} = rx_t + c,$$

with initial condition x_0 can be written in several different but equivalent forms as

$$\begin{aligned} x_t &= r^t x_0 + c(1 + r + r^2 + r^3 + \dots r^{t-1}) \\ &= r^t x_0 + \frac{r^t - 1}{r - 1} c \\ &= r^t \left(x_0 - \frac{c}{1-r} \right) + \frac{c}{1-r} \end{aligned}$$

Note: The second and third form are only valid when $r \neq 1$. We can write things even simpler by setting $x^* = \frac{c}{1-r}$. Then the general solution is

$$x_t = r^t (x_0 - x^*) + x^*$$

Observation:

1. Suppose that you have an updating function, but you can only observe your system every other generation. Then $x_{t+2} = f(x_{t+1}) = f(f(x_t)) = (f \circ f)(x_t)$. Hence the composition of updating functions corresponds to an observation two generations ahead.
2. Suppose that you missed an observation and want to infer the state in the previous generation from the next. Then $x_t = f^{-1}(x_{t+1})$ if the inverse function exists. Hence, the inverse updating function corresponds to going backwards in time.

3.1 Practice makes progress

This material is in section 3.1 of the book (2.1 in the first edition). Suggested exercises are

- Drill questions: 1-14, 21-24, 25-27
- Applications: 32, 33, 36, 37, 41-45
- Advanced: 58-61

Question 1: Suppose someone has three drinks of alcohol that bring the alcohol content in their body to 42 grams. Then the person stops drinking. Each hour, 45% of the alcohol are eliminated from the body.

1. Write the DTDS for the amount of alcohol in the body on an hourly basis.
2. Identify the initial condition and give the general solution.
3. If the amount of alcohol in the body has to be below 8 grams before one can drive, how long does the person have to wait before they can drive?

Question 2: Assume that the dynamics of caffeine absorptions are given by $C_{t+1} = 0.87C_t$, where t is time in hours and C_t is the concentration of caffeine. If $C_0 = 1000$, estimate the time needed for 80% of the caffeine to be eliminated from the body (i.e. 20% left).

Question 3: A population of rabbits is growing at a rate of 10% per year. Write down the discrete dynamical system that describes the evolution of the rabbit population. If the population is initially 1000, how many years will it take for the population to exceed 100000?

Question 4: A city is growing at a rate of 1% per year and initially has 1,000,000 individuals. What will be the population in 50 years' time?

4 DTDS - part II

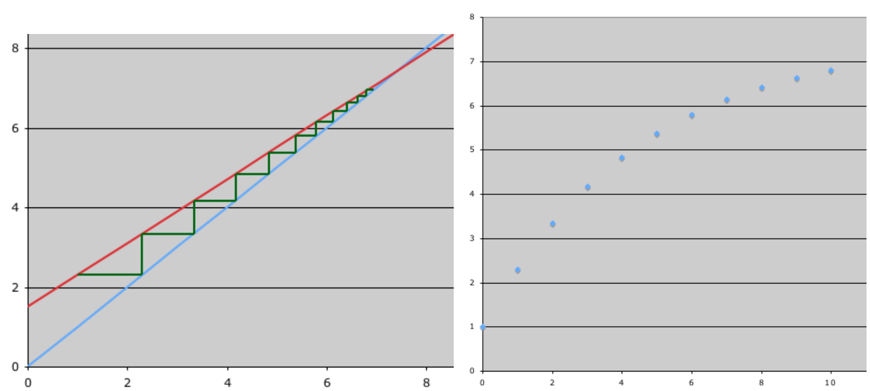
GOAL: Visualize the behavior of a DTDS, identify special points

Graphing I: Once we have calculated a solution, we can graph it as a sequence of points $(0, x_0)$, $(1, x_1)$, $(2, x_2)$,... (t, x_t),....

Graphing II: *Without* calculating a solution, we can visualize a solution of a DTDS by using a technique called **cobwebbing**. The recipe for cobwebbing is as follows.

1. Graph the updating function.
2. Starting from $(x_0, 0)$, draw a line *vertically* to the graph of the updating function.
3. Starting from the point $(x_0, f(x_0))$ on the graph of the updating function, draw a line *horizontally* to the diagonal.
4. Repeat the previous two instructions
5. In a second plot with aligned horizontal axes, record the points (t, x_t).

Note: Use the excel files available in the course material to generate your cobwebbing. The two plots below were generated using `LinearDTDS.xls` with updating function $f(x) = 0.8x + 1.5$ and initial condition $x_0 = 1$. The plot on the left shows the cobweb (updating function in red, diagonal in blue, cobweb in green), the plot on the right the solution as a function of time.



Definition: A constant solution of a DTDS is called a *fixed point* or *steady state* or *equilibrium*. A fixed point is where the updating function intersects the diagonal. To calculate a fixed point algebraically, one solves the equation $x = f(x)$ for x.

Example: For the linear DTDS $x_{t+1} = rx_t + c$ the fixed point is $x^* = \frac{c}{1-r}$, when $r \neq 1$.

4.1 Practice makes progress

This material is in section 3.2 of the book (2.2 in the first edition). Suggested exercises are

- second edition: 3.1: 17-20; 3.2: 3-12, 17-22, 31-34
- first edition: 2.1: 15-18; 2.2: 3-12, 13-16, 17-34

Question 1: A population of butterflies lives on a meadow, surrounded by forest. We want to investigate the dynamics of the population. We denote the number of butterflies at the beginning of season t by x_t. Over the course of a season, 30% of the butterflies that were there at the beginning die. During each season, 20 new butterflies arrive from other meadows.

1. Write the DTDS for the number of butterflies. What is the updating function?
2. Starting with 40 butterflies in season 0, calculate their number in seasons 1, 2, 3.
3. Calculate the fixed point of the DTDS.
4. Write the solution of the DTDS in terms of a general initial condition x_0.
5. Draw the cobweb for this DTDS, starting at $x_0 = 40$. Also draw the solution as a function of time.
6. Suppose that through some conservation measures, we can improve the quality of the pond and reduce the death rate of the butterflies. To which level do we have to reduce the death rate if we want the steady state butterfly population to be 100?

Question 2: Consider the nonlinear DTDS $x_{t+1} = \frac{rx_t}{1+0.2x_t}$, where $r > 0$ is some parameter.

1. Calculate the fixed point(s). (Note: the answer may contain parameter r.)
2. Now set $r = 3$. Calculate x_1, x_2, x_3 starting form $x_0 = 8$.
3. Keeping $r = 3$, calculate x_1, x_2, x_3 starting form $x_0 = 12$.
4. Do you observe a trend in these values? If so, what is it?

Question 3: In a forest in Alberta, every year 20% of the population of red deer either die of natural causes or are eaten by predators. In the meantime, there are 1000 new red deer. The discrete-time dynamical system that gives the population of red deer each year is $p_{t+1} = 0.8p_t + 1000$.

1. If there are 2000 red deer now, how many red deers will be there three years later?
2. Give the updating function of the dynamical system. Find its inverse, if it exists.
3. Determine all equilibrium points of the dynamical system.
4. Find the general solution of the dynamical system (ie., a formula in terms of t) given the initial condition $p_0 = 2000$.

5. Draw the solution of the dynamical system with $p_0 = 2000$ (four points are enough).
6. Draw the cobweb diagram of the dynamical system with $p_0 = 2000$ (four iterations are enough).

Question 4: Consider the discrete-time dynamical system $N_{t+1} = \frac{rN_t}{N_t - 3}$, where r is a parameter. Find all values (if they exist) of parameter r, for which the system has (i) no equilibria, (ii) one equilibrium, (iii) nonnegative equilibria.

Question 5: For which value(s) of r does the DTDS $N_{t+1} = \frac{rN_t}{1+N_t}$ have a positive equilibrium?

Question 6: Suppose you deposit \$1000 each week into a special savings account, but the bank takes 5% of the total in fees. A discrete-time system modelling your investment is $x_{n+1} = 0.95x_n + 1000$.

1. If you initially have \$1500 in the bank, how much money will you have after the third week?
2. Write down the updating function of the dynamical system.
3. Find all equilbrium points of the dynamical system.
4. Give the solution of the dynamical system with $x_0 = 1500$.
5. Draw the solution of the dynamical system with $x_0 = 1500$. (Four points are enough.)
6. Draw the cobweb diagram of the dynamical system with $x_0 = 1500$. (Four iterations are enough.)

Question 7: A disease is spreading through campus. Each day, the number of people infected depends on how many were infected the day before, according to the formula $y_{n+1} = \frac{6.5y_n}{1+0.01y_n}$.

1. If one person is initially infected, how many people are infected after the first day? after the second day? after the third?
2. Find all equilibrium points of the dynamical system.
3. How many people would you guess will be infected eventually?

Question 8: Suppose that every morning a patient receives the same dose of drug. From the dose, the drug concentration in his blood increases by 2. Over the course of 24 hours between doses, 75% of the drug in the blood is removed.

1. Write the linear DTDS for the drug concentration, $x_{t+1} = f(x_t)$, and find x_4 when $x_0 = 88$.
2. Draw the updating function and start the cobwebbing process at $x = 0.2$.

3. Find the fixed point explicitly.

Question 9: In order to keep the songbirds in the back yard happy, one person puts out 20g of seeds at the end of each week. During the week, the birds find and eat 2/3 or the available seeds. The DTDS for the amount of seeds in the back yard is $S_{t+1} = 1/3S_t + 20$, where t is measured in weeks and seeds are counted just before a new supply is provided.

1. What is the updating function of the DTDS?
2. Find the fixed point of the DTDS if there is one.
3. Find the general solution formula for the DTDS, i.e., $S_t = ...$
4. Graph the updating function and draw the cobwebbing, starting from $S_0 = 5$, for at least 4 steps.

Question 10: A group of patients is given a certain dose of a drug once. Two measurements of concentration of the drug in the blood are taken 24 hours apart to determine the rate at which the drug is removed from the blood stream. The measurements are given below.

patient	initial measurement	final measurement
1	3	1
2	4.5	1.5
3	0.6	0.2
4	1.8	0.6

1. Write a DTDS of the form $x_{t+1} = ax_t$ for drug removal and find the value of a.
2. For patient 1, how long will it take until the drug concentration is below 0.1?
3. How long does it take for the initial concentration to decrease by 50%?
4. Now patients are given a dose every 24 hours, i.e., we have the DTDS $x_{t+1} = ax_t + b$ with a as in part (a). How much of the drug has to be given so that the steady state concentration is 6?

5 DTDS - part III

GOAL: Analyze DTDS, and learn more complicated examples.

Examples: Cobweb a linear DTDS with (i) $x_{t+1} = 1.5x_t$, (ii) $x_{t+1} = 0.5x_t$, (iii) $x_{t+1} = -0.5x_t$, (iv) $x_{t+1} = -1.5x_t$. Notice that in cases (ii) and (iii) all solutions approach the only steady state $x^* = 0$, whereas in cases (i) and (iv), solutions grow in absolute value.

Definition: A fixed point is called *stable* if all nearby solutions approach the point, and *unstable* if at least one nearby solution does not approach the point.

Obervation: In the linear DTDS $x_{t+1} = rx_t + c$, the solution formula

$$x_t = r^t(x_0 - x^*) + x^*$$

makes it clear that x_t approaches x_0 exactly when $|r| < 1$. So we say that the fixed point x^* is stable exactly when $|r| < 1$. There are four qualitatively different cases for cobwebbing according to (i) $0 < r < 1$, (ii) $r > 1$, (iii) $-1 < r < 0$, and (iv) $r < -1$.

Note: A stable fixed point gives information about the long-term behavior of a DTDS. An unstable fixed point would not be observed in nature, but still carries important information.

Nonlinear updating functions Most processes in nature are not linear. Therefore, we need nonlinear updating functions to describe such processes. There is in general no explicit way to write down a general solution. But cobwebbing works. In each of the following examples, find steady states analytically and use cobwebbing to determine their stability.

1. Beverton-Holt updating function $f(x) = \frac{rx}{1+\alpha x}$
2. Ricker updating function $f(x) = rx\exp(-\alpha x)$
3. Allee updating function $f(x) = \frac{ax^2}{b^2+x^2}$
4. Alcohol adsorption (see book)
5. Heartbeat model (see book)

5.1 Practice makes progress

This material is in sections 3.2-3.4 of the book (2.2, 2.3 and 2.5 in the first edition). Suggested exercises are

- second edition: **3.2:** 13-16, 23-30, 35-40; **3.3:** 1-19, 27-29, 30-34; **3.4:** 11-18, 39-42
- first edition: **2.2:** 13-16 **2.3:** 1-19, 27-29, 28-32; **2.5:** 11-18, 39-42

Question 1: Apply the stability criterion to all the practice problems with linear updating function in the previous section and find whether the steady state is stable.

Question 2: Consider the discrete-time dynamical system (DTDS) $M_{t+1} = -0.8M_t + 6$.

- Find the updating function of the DTDS.
- (b) Find the equilibrium point of the DTDS.
- Give the general solution formula for the DTDS.
- Calculate M_{10} if $M_0 = 0$.
- Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.
- Is the equilibrium point stable or unstable?

Question 3:

1. Because of high mortality and low reproductive success some fish species experience exponential decline over many years. Atlantic Salmon in Lake Ontario, for example, declined by 80% in the 20-year period leading up to 1896. Denote the number of Atlantic Salmon in Lake Ontario in year t by x_t and write the equation $x_{t+1} = rx_t$. Calculate the value of r.

2. Due to fishing restrictions, the value of r changed and the population is less at risk now. The major reason for the recovery of Atlantic Salmon, however, is a massive restocking program in Ontario. The population dynamics can now be described by the DTDS $x_{t+1} = 0.2x_t + c$, where c is the number of fish restocked every year.
   - What is the updating function of this DTDS ?
   - What is the equilibrium point of this DTDS ?
   - Is the equilibrium point stable or unstable for the DTDS? Why?

3. Now we assume that there are 1000 fish restocked annually.
   - Find the general solution formula for the DTDS with this value.
   - Draw the cobweb, starting from $M_0 = 1000$ for at least 4 steps.
   - How does the number of restocked fish need to be adjusted to ensure an equilibrium population of 1500 fish?

Question 4: A patient receives a daily dose $d = 5$mg of the drug FilGud®. In the course of 24 hours, 60% of the drug is absorbed and a fraction of $p = 0.4$ remains in the blood. The DTDS modelling the daily concentration M_t of FilGud® in the blood immediately after administering the dose is $M_{t+1} = pM_t + d = 0.4M_t + 5$.

- Find the updating function of the DTDS.
- Find the equilibrium point of the DTDS.
- Give the general solution for the DTDS with general initial condition M_0.
- Calculate M_5 if $M_0 = 0$.
- Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.
- Is the equilibrium point stable or unstable?
- Due to sudden complications, the patient now also needs to take the drug WelSun®. This drug inhibits the uptake of FilGud® so that only 50% will be absorbed and a fraction of $\tilde{p} = 0.5$ will remain in the blood. Calculate the new daily dose $\tilde{d}$ of FilGud® needed to maintain the equilibrium concentration of that drug at the same level as before.

6 Limits of functions

GOAL: Characterize the behavior of a function near a point where the function might not be defined.

Definition: Limit of a function. We say that the limit of a function f as x approaches a equals L and write

$$\lim_{x\to a} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x very close to a.

Observations:

- The value of $f(a)$, if it even exists, does not matter in the definition of a limit.
- We can define one-sided limits in a similar way and denote them by $\lim_{x\to a^+}$ and $\lim_{x\to a^-}$.

Definition: Existence of a limit. We say that the limit of f as x approaches a exists if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Observations:

- Calculators might fail at finding limits. In this class, we use other means.
- Identify limits when a graph is given.
- Algebraic rules to find limits.

A combination of the following rules together with algebraic manipulations of the expressions allow us to calculate many limits.
Limit laws: The following are true.

1. $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$.

If the limits involved exist, then the following hold.

2. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x) \neq 0$

Direct Substitution Rule: If the function is a polynomial, a rational function, a trig function, an exponential or logarithm function, or a root function, or a composition of these, and if a is in the domain of the function, then $\lim_{x\to a} f(x) = f(a)$, in particular the limit exists.

6.1 Practice makes progress

The material is covered in Section 4.2 of the book (3.2 in the first edition) with Section 4.1 (3.1) being one motivation to study such things as limits. Suggested exercises are

- second edition: **4.2:** 1–7, 10–13, 30–55
- first edition: **3.2:** 1–7, 10–13, 30–55

Question 1: Does the limit $\lim_{x\to 4} \dfrac{x-\sqrt{3x+4}}{4-x}$ exist? If so, what is its value?

Question 2: Let $G(x) = \frac{x^2-5x+6}{|3-x|} + |x-2|$.
a) Find $\lim_{x\to 3^+} G(x)$.
b) Find $\lim_{x\to 3^-} G(x)$.
c) Does $\lim_{x\to 3} G(x)$ exist?

Question 3: For a real number b, consider the function

$$f(x) = \begin{cases} \sin(x-b), & x>0 \\ x^2+1, & x<0. \end{cases}$$

Find the smallest possible positive value of b such that the limit $\lim_{x\to 0} f(x)$ exists.

Question 4: Use your calculator to guess whether the limit $\lim_{x\to 0}\left(\dfrac{e^x-1}{x}\right)$ exists and what its value might be. (Yes, this is one of the few problems where you are allowed to use a calculator. The point is that you remember to choose values on both sides of the point $x=0$ in your sequences.)

Question 5: Calculate the following limits, if they exist, or explain why they do not exist. Justify your answers without using sequences of numerical values for x.

- $\lim_{x\to 1} \dfrac{1-x^2}{x^2-3x+2}$.
- $\lim_{x\to 0} \dfrac{\sin^2(x)}{\cos^2(x)}$
- $\lim_{t\to 0} \left(\dfrac{1}{t\sqrt{1+5t}} - \dfrac{1}{t}\right)$
- $\lim_{t\to 0} \left(\dfrac{t}{1-\sqrt{1+t}}\right)$

Question 6: Let g be defined by $g(x) = \dfrac{x^2-9}{|x-3|}$.
a) Calculate $\lim_{x\to 3^+} g(x)$.

b) Calculate $\lim_{x\to 3^-} g(x)$.
c) Does the limit $\lim_{x\to 3} g(x)$ exist?

Question 7: Let g be defined by $g(x) = \dfrac{x^2-1}{|x-1|}$.
a) Calculate $\lim_{x\to 1^+} g(x)$.
b) Calculate $\lim_{x\to 1^-} g(x)$.
c) Does the limit $\lim_{x\to 1} g(x)$ exist?

Question 8: For which number a does the limit $\lim_{x\to 0} f(x)$ exist, where

$$f(x) = \begin{cases} e^{x-a} - 1, & x > 0 \\ x^2 + 1, & x < 0. \end{cases}$$

Question 9: (a) Find the following limit without using a calculator

$$\lim_{x\to -2} \frac{|x+3|-1}{x^2-4}$$

(b) Does the following limit exist? If yes, give the limit, if not, justify your answer. If you need a calculator to work out the answer, give at least 4 values of x that you tried.

$$\lim_{x\to 2} \frac{|x+3|-1}{x^2-4}$$

6.2 Limits of functions – Lecture

GOAL: Characterize the behavior of a function near a point where the function might not be defined.

Motivation example: Characterize the slope of a function as the limit of the slope of secant lines. Pick $g(x) = x^3$. Then the slope of the secant line between points x and y is

$$f(x) = \frac{g(x) - g(y)}{x - y}, \qquad x \neq y.$$

We would like to think of the slope of the function as the slope of the tangent line, which is the value of the function f when $x = y$. But the function is not defined for $x = y$, exactly where we are most interested in: when $x = y$.

When you don't know what to do, sometimes it helps to try things and play with them. So, let's fix $y = 1$ and use a calculator to see what happens when x is close to 1 in the function $f(x) = \frac{x^3-1}{x-1}$.

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	2.79	2.97	2.997	3.3	3.03	3.003

It looks like the closer we choose x to 1, the closer $g(x)$ will be to 3. We formalize this terminology.

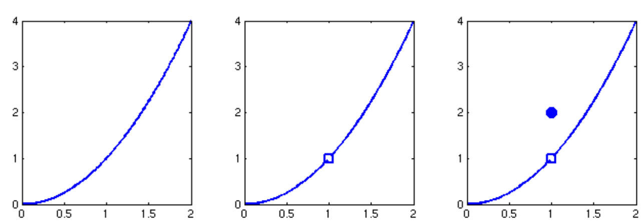
Definition: Limit of a function. We say that the limit of a function f as x approaches a equals L and write

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x very close to a.

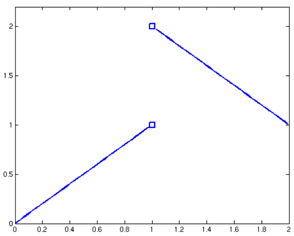
So, back to the example above, we are tempted to say that $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$. But we need to be careful. Limits are sometimes tricky and sometimes deceiving. Let's start slowly.

Observation: The value of $f(a)$, if it even exists, does not matter in the definition of a limit. The plots below show three functions that only differ in their value at point a: one has value 1, one is not defined and one has value 2 there. Yet, the limit is the same for all three: $\lim_{x \to a} f(x) = 1$.



Example: Take a look at the graph of the function below. No matter how we choose a value L, we can never satisfy the conditions so that $\lim_{x\to 1} f(x)$ would equal L. But we think that if we allowed x to approach 1 from only one side, we would get a limit.

$$f(x) = \begin{cases} x, & x < 1 \\ 3 - x, & x > 1. \end{cases}$$



Definition: One-sided limit. We say that the limit of a function f as x approaches a *from above (below)* equals L and write

$$\lim_{x\to a^+} f(x) = L, \quad \text{or} \quad \lim_{x\to a^-} f(x) = L,$$

respectively, if we can make $f(x)$ as close to L as we wish by choosing x very close to a and larger (smaller) than a.

Definition: Existence of a limit. We say that the limit of f as x approaches a exists if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Example: The limit $\lim_{x\to 0}\sqrt{x}$ does not exist, since the function is not even defined for $x < 0$. The one-sided limit $\lim_{x\to 0^+}\sqrt{x} = 0$ does exist as does the limit $\lim_{x\to 0}\sqrt{|x|} = 0$.

HOW CAN WE EVALUATE LIMITS?

- by calculator: this can go wrong (see below) and is generally not accepted in this course
- by reading the graph (as above): this requires having a graph, and graphing is one of the things that we will learn later in this course.
- by limit laws and algebraic manipulations: this is the most elegant and only precise way, the one we learn in this course.

First example for how a calculator may fail: This example is to caution you against the simple use of calculators when evaluating limits. First of all, if you use a calculator, you need to choose values of x close to a, larger and smaller than a. How this approach might fail miserably? Let's look at the function

$$f(x) = \frac{\sqrt{x^6 + 25} - 5}{x^6}, \qquad a = 0.$$

The calculator approach leads us to guess that in the limit as $x \to 0$ the function approaches 0. But the actual value is 0.1 (as we will see later). So, let's come up with a catalogue of functions for which it is easy to find limits.

Second example for how a calculator may fail: Now try to find the limit

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right).$$

If we choose for x the values 1, 1/10, 1/100, 1/1000,..., the calculator should give a value of 0 (some don't because of rounding errors). However, if we choose for x the values 2/1, 2/11, 2/111, 2/1111,..., then the calculator gives -1 all the time. We have two different candidates for limits, and no way to tell since we cannot evaluate *every possible* way in which x could approach zero.

Limit laws that never fail: The following are true.

1. $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = a$.

If the limits involved exist, then the following hold.

2. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x) \neq 0$

Example: Let's use these rules to calculate the limit $\lim_{x\to 2}(x^3-3x+5)$. According to the sum rule, we can write this as

$$\lim_{x\to 2}(x^3-3x+5) = \lim_{x\to 2}(x^3) + \lim_{x\to 2}(-3x) + \lim_{x\to 2} 5.$$

The we use the product rule (note that the factor -3 is a constant function) and write

$$\lim_{x\to 2}(x^3-3x+5) = (\lim_{x\to 2} x)^3 - 3\lim_{x\to 2} x + \lim_{x\to 2} 5.$$

Each term is now in the form or the first rule above. Hence the limit exists and is given by

$$\lim_{x\to 2}(x^3-3x+5) = (2)^3 - 3\cdot 2 + 5 = 7.$$

Direct Substitution Rule: The previous example is a special case of the direct substitution rule: If the function is a polynomial, a rational function, a trig function, an exponential or logarithm function, or a root function, or a composition of these, and if a is in the domain of the function, then $\lim_{x\to a} f(x) = f(a)$, in particular the limit exists.

Examples: The limit of the rational function

$$\lim_{x\to 1} \frac{2x^2+7x-1}{3x+2}$$

can be evaluated with the direct substitution rule. Since the denominator is not zero for $x = 1$ we find the limit by substitution; it equals 2.

This rule is very helpful in many cases, but the interesting cases are those when the rule cannot be applied. For example, our motivation example at the beginning. In those cases, we need to remember that $\lim_{x\to a}$ means that x is close to a but *not equal* to a. So, we can use any algebraic manipulations that are valid when $x \neq a$ and transform the function whose limit we want to find into a function that is in the list above where the direct substitution rule applies.

Examples:

1. For the motivation example, we use long division and then direct substitution
$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3.$$

2. For the example where the calculator failed, we rationalize the numerator and then use direct substitution
$$\lim_{x\to 0} \frac{\sqrt{x^6+25}-5}{x^6} = \lim_{x\to 0} \frac{1}{\sqrt{x^6+25}+5} = \frac{1}{10}.$$

3. There are other ways to simplify first. For example
$$\lim_{x\to 0} \frac{(2+x)^2-4}{x} = \lim_{x\to 0} \frac{x^2+4x+4-4}{x} = \lim_{x\to 0} \frac{x(x+4)}{x} = \lim_{x\to 0} x+4 = 4.$$

4. Quite similar is the following
$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$$

5. The sign-function can be written using the absolute value as $\mathrm{sgn}(x) = \frac{x}{|x|}$ when $x \neq 0$ and $\mathrm{sgn}(x) = 0$. In this case, we have to use one-sided limits to understand what happens as $x \to 0$.
$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1, \quad \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^+} (-1) = -1.$$

   Direct substitution gives $\mathrm{sgn}(0) = 0$. The one-sided limits exist, but they do not agree, and hence the limit does not exist.

7 Limits, infinity, and continuity

GOAL: Extend the material from the previous class to infinity; introduce continuity.

Definition: Infinite limits. We say that the limit of a function f as x approaches a is infinity (or negative infinity) if we can make $f(x)$ as large (negative and large) as we wish by choosing x very close to a. We write

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to a} f(x) = -\infty.$$

This definition applies also to one-sided limits. Geometrically, an infinite limit at a finite value a corresponds to a *vertical asymptote* of the graph at $x = a$.

NOTE: Infinity is *not* a number.

We have relatively few tools to prove infinite limits. Examples are rational functions; and remember that if $g(x) \to 0$ as $x \to a$ then $1/g(x) \to \pm\infty$ as $x \to a$.

Definition: Limits at infinity We say that the limit of a function f as x approaches ∞ equals L and write

$$\lim_{x\to\infty} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x arbitrarily large. We can also consider the limit $x \to -\infty$, but obviously, these limit can only be one-sided. A limit at infinity corresponds to a *horizontal asymptote* of the graph at $y = L$.

These limits at infinity can be evaluated with the same algebraic procedures as limits at finite values a.

Definition: Continuity A function f is called *continuous at a point a* if $f(a)$ exists, and if the limit equals that value, $\lim_{x\to a} f(x) = f(a)$. In particular, this limit needs to exist. A function is called *continuous on an interval* if it is continuous at every point in that interval.

Examples: polynomials, rational functions (where defined), trigonometric and inverse trigonometric functions (where defined), exponentials and logarithms are continuous. The sign function is not continuous at $x = 0$.

Note: Look out for when checking continuity: division by zero, log of zero, piecewise defined functions at places where the function definition changes.

Theorem: Continuity and exchanging limits. If f is continuous and g is a function so that $\lim_{x\to a} g(x) = b$ exists, then we can take the limit of the composition

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

7.1 Practice makes progress

The material is covered in Sections 4.3/4.4 of the book (3.3/3.4 in the first edition). Please note that we are not covering "Comparing functions at infinity" (this topic will come back to us later) not will we talk about "Limits of sequences". In the section on continuity, we place little emphasis on questions of "input/output precision". Suggested exercises are

- second edition: **4.3:** 1–14, 18–26, 27–30, 31–50
- second edition: **4.4:** 1–29, 24–37
- first edition: **3.3:** 1–14, 18–24, 25–48, 50, 51
- first edition: **3.2:** 1–29, 24–37

Question 1: Calculate the following limits:

$$\lim_{x\to 2}\frac{x^2-\frac{3}{2}x-1}{x-2} \qquad \lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+2}} \qquad \lim_{x\to 0}\frac{1}{x}\left(\frac{1-x}{1+x}\right)$$

$$\lim_{x\to 1}\frac{1-x^2}{x^2-3x+2} \qquad \lim_{x\to 0}\frac{\sin^2(x)}{\cos^2(x)} \qquad \lim_{t\to 0}\left(\frac{1}{t\sqrt{1+5t}}-\frac{1}{t}\right)$$

Question 2: Does the limit $\lim\limits_{x\to-\infty} e^{1/x}$ exist? If so, what is its value?

Question 3: Consider the function $f(x)=\frac{1+2e^{-x}}{1-e^{-x}}$.
(a) Find $\lim_{x\to\infty} f(x)$.
(b) Find $\lim_{x\to-\infty} f(x)$.
(c) Are there any infinite limits? If yes, find the left and right hand limit in each case.

Question 4: Let $f(x)=\dfrac{x^2-4}{|x-2|}$. Calculate $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ Is f continuous at $x=2$?

Question 5: Is the following function continuous at $x=1$? Justify your answer in a short sentence.

$$f(x)=\cos(2x)+\frac{3x^2-5x}{x^2-2}$$

Question 6: Can one choose a value for a such that the following function is continuous at $x=3$? If yes, what is the value and why? If no, why not?

$$f(x)=\begin{cases}\frac{|x-3|}{x^2-9}, & x\neq 3\\ x+a, & x=3.\end{cases}$$

Question 7: Let $f(x) = \begin{cases} \frac{x^2-4x+3}{(x-1)^3} & \text{if } x \neq \pm 1, \\ x+b & \text{if } x = \pm 1. \end{cases}$, with b a real number

(a) Find $\lim_{x\to 1} f(x)$.
(b) Is there a value of b that makes the function f continuous at $x = 1$?
(c) Find $\lim_{x\to -1} f(x)$.
(d) Is there a value of b that makes the function f continuous at $x = -1$? If yes, then provide the value.

Question 8: For what values of a and b is the following function continuous everywhere?

$$f(x) = \begin{cases} a\sin(x) + b, & x \leq 0 \\ x^2 + a, & 0 < x \leq 1 \\ b\cos(2\pi x) + a, & x > 1 \end{cases}$$

Question 9: Find the value of a so that the following function is continuous

$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ a & x = 1. \end{cases}$$

Question 10: Consider the function $g(x) = \begin{cases} \frac{a}{(\sin(x))^2+1} & \text{if } x < \frac{\pi}{2} \\ \frac{kx+1}{x+1} & \text{if } x \geq \frac{\pi}{2} \end{cases}$

(a) What is the condition on a and k such that g is continuous at $\pi/2$?
(b) Find a and k so that g is continuous and has the horizontal asymptote $y = 2$ as $x \to \infty$.

7.2 Limits, infinity and continuity – lecture

GOAL: Extend the material from the previous class to infinity; introduce continuity.

Definition: Infinite limits. We say that the limit of a function f as x approaches a is infinity (or negative infinity) if we can make $f(x)$ as large (negative and large) as we wish by choosing x very close to a. We write

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \lim_{x\to a} f(x) = -\infty.$$

This definition applies also to one-sided limits. Geometrically, an infinite limit at a finite value a corresponds to a *vertical asymptote* of the graph at $x = a$.

Examples:

1. $f(x) = \frac{1}{x^2}$ and $a = 0$. The function is not defined at 0, and as $x \to 0$ we have a fixed positive number divided by smaller and smaller numbers. As a result, the limit is infinite. For example, if we want to make $f(x)$ larger than 10,000 then we have to make $|x| < 0.01$.

2. $f(x) = \frac{1}{x}$ and $a = 0$. Again, the function is not defined at 0, but we need to be careful with signs. When $x > 0$ the fraction is positive and gets arbitrarily large, when $x < 0$ it is negative and large in absolute value. Hence we have one-sided limits only. For example, if we want to make $f(x)$ larger than 1000, then we have to have $0 < x < 0.001$. To make $f(x) < 1000$ we have to choose $-0.001 < x < 0$.

3. All rational functions work similarly. When the denominator approaches zero and the numerator does not, then we need to check signs to see whether and from which side the function approaches $\pm\infty$.

4. Many more functions have similar properties, for example $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ is a fraction of functions. The zeros of the denominator are not zeros of the numerator, hence there are infinite limits at those places.

5. Another class of functions with infinite limits are logarithms, e.g., $f(x) = \ln(x)$ as $x \to 0$. This time, only the one-sided limit exists.

Definition: Limits at infinity. We say that the limit of a function f as x approaches ∞ equals L and write

$$\lim_{x\to\infty} f(x) = L$$

if we can make $f(x)$ as close to L as we wish by choosing x arbitrarily large. We can also consider the limit $x \to -\infty$, but obviously, these limit can only be one-sided. A limit at infinity corresponds to a *horizontal asymptote* of the graph at $y = L$.

Examples:

1. The fraction $f(x) = 1/x$ is an important example. As $x \pm \infty$, the fraction approaches zero. More generally, if $g(x)$ can grow arbitrarily large, then $1/g(x)$ will grow arbitrarily small. For example, to make $f(x)$ smaller than 0,01, we have to choose $x > 100$.

2. Given a rational function, we divide numerator and denominator by the highest power of x, the we see what happens. For example

$$\lim_{x\to\infty} \frac{ax}{b+x} = \lim_{x\to\infty} \frac{\frac{ax}{x}}{\frac{b+x}{x}} = \lim_{x\to\infty} \frac{a}{b/x+1} = a$$

and

$$\lim_{x\to\infty} \frac{ax}{b+x^2} = \lim_{x\to\infty} \frac{\frac{ax}{x^2}}{\frac{b+x^2}{x^2}} = \lim_{x\to\infty} \frac{a/x}{b/x^2+1} = 0.$$

Note that for rational functions $r(x)$, if $\lim_{x\to\infty} r(x) = L$ then also $\lim_{x\to-\infty} r(x) = L$.

3. $f(x) = e^{ax}$ with $a > 0$ has $\lim_{x\to\infty} f(x) = \infty$ but $\lim_{x\to-\infty} f(x) = 0$.

4. Recall the arctan function, the inverse function of tan? Its horizontal asymptotes are $\pm\pi/2$. Why? What are the vertical asymptotes of tan? And how are a function and its inverse related?

5. We can build fractions of functions other than polynomials, and those functions may have different limits as $x \to \pm\infty$. The idea of dividing by the "highest power" still works. For example

$$\lim_{x\to\infty} \frac{e^x - 1}{5 + 2e^x} = \lim_{x\to\infty} \frac{\frac{e^x-1}{e^x}}{\frac{5+2e^x}{e^x}} = \lim_{x\to\infty} \frac{1 - e^{-x}}{5e^{-x}+2} = 1/2.$$

On the other hand, $\lim_{x\to-\infty} \frac{e^x-1}{5+2e^x} = -1/5$.

We have seen that the direct substitution rule often allows us to evaluate limits fairly easily and that many functions allow us to use this rule. We now give these functions a special name: continuous functions.

Definition: Continuity. A function f is called *continuous at a point* a if $f(a)$ exists, and if the limit $\lim_{x\to a} f(x) = f(a)$. In particular, this limit needs to exist. A function is called *continuous on an interval* if it is continuous at every point in that interval. Roughly, a function is continuous if it does not jump. If a function is continuous everywhere where it is defined, we say that it is continuous on its domain. If the context is clear, we simply say continuous.

Note: If you know that a certain function is continuous, then limits (within the domain of the function) are easily evaluated by the direct substitution rule.

Examples:

1. The following functions are continuous: constant, linear, exponential, logarithm, root, absolute value, sin, cos.
2. If f, g are continuous, then so are $f \pm g$, $f \cdot g$ f/g (provided $g \neq 0$) and $f \circ g$.
3. Therefore, all polynomials, rational functions (where defined), trigonometric and inverse trigonometric functions (where defined) are continuous.
4. The sign function is not continuous at $x = 0$. (There is nothing wrong about discontinuous functions. Many processes in real life are best desctibed by discontinuous functions. For example: harvesting, drug levels in body...)

Note: Look out for when checking continuity: division by zero, log or zero, piecewise defined functions at places where the function definition changes.

Example: Find c such that the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} x + c, & x < 0 \\ \cos(x), & x \geq 0. \end{cases}$$

Theorem: Continuity and exchanging limits. If f is continuous and g is a function so that $\lim_{x\to a} g(x) = b$ exists, then we can take the limit of the composition

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Example: Find the limit $\lim_{x\to\infty} e^{1/x}$.

8 Differentiability

GOAL: Define the slope of a function at a point.

Idea: We know what the slope of a line is. Given two points, $x \neq y$ in the domain of function f, we write the slope of the secant line though the two points as $\frac{f(y)-f(x)}{y-x}$. Now we take the limit as $y \to x$ and define this limit to be the slope of the tangent line.

Definition: A function f is called *differentiable at a point* x if the limit

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$$

exists. (Note that f has to be defined and continuous at x.) A function is called differentiable on an open interval, if it is differentiable at each point of the interval.

Note: Failure to be differentiable can correspond to a corner, a cusp, a vertical asymptote, a doscontinuity.

Derivatives and Graphs: We have a correspondence

$$f'(x) > 0 \Leftrightarrow f \text{ increasing at } x$$
$$f'(x) = 0 \Leftrightarrow f \text{ has horizontal tangent at } x$$
$$f'(x) < 0 \Leftrightarrow f \text{ decreasing at } x$$

Definition: x is called a *critical point of* f if x is in the domain of f and either $f'(x) = 0$ or $f'(x)$ is not defined.

Note: Calculation of derivative from the definition/from first principles can be done in simple cases. But in general, we want to have rules that we can use faster, rather than going back to the definition every time.

Differentiation rules: If the derivatives exist, then the following rules hold

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
2. $(f \pm g)'(x) = f'(x) \pm g'(x)$
3. If $h(x) = f(x) \cdot g(x)$ then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. If $h(x) = \frac{u(x)}{v(x)}$ then $h'(x) = \frac{u'(x)v(x)-u(x)v'(x)}{v^2(x)}$

8.1 Practice makes progress

The material is covered in Sections 4.5/5.1/5.2 of the book (3.5/4.1/4.2 in the first edition). The second edition covers the derivative of the exponential function already here. The first edition has it later. We will also treat it later. Suggested exercises are

- second edition: **4.5:** 8, 9, 15–17, 22–28, 39–43, 44–49, 50–53, 54–57
- second edition: **5.1:** 1–42, **5.2:** 1–43 (*Not* the problems with the exponential function)
- first edition: **3.5:** 8, 9, 15–17, 22–28, 39–43, 44–49, 50–53, 54–57
- first edition: **4.1:** 1–16, **4.2:** 1–19

Question 1: Use the definition of the derivative to calculate the derivative of the functions.

$$\text{(a) } f(x) = 1 + \sqrt{2x+3} \qquad \text{(b) } g(x) = \frac{x}{1+x} \qquad \text{(c) } h(x) = \sqrt{x^2-1}$$

$$\text{(d) } f(x) = \frac{2}{2015-x} \qquad \text{(e) } g(y) = \frac{2}{3+4y} \qquad \text{(c) } h(z) = (z-1)^2 + 3z - 12$$

Question 2: Use the rules from class to calculate the derivatives of the following functions.

$$\text{(a) } g(x) = \frac{1+x^2}{\sqrt{x}+x^{-1}} \qquad \text{(b) } f(x) = 12x^{12} - 11x^{11} + 10x^{10}$$

$$\text{(c) } h(z) = \frac{1}{z^3} + \frac{1}{z^{1/3}} \qquad \text{(d) } f(x) = 15y^2 - 3y + 1$$

Question 3: Give an example of a function that is continuous everywhere but not differentiable at $x = 2$.

Question 4 (hard!): Consider the function

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x^2, & x \neq \mathbb{Q}. \end{cases}$$

Is this function continuous at $x = 0$? Is it differentiable at $x = 0$?

8.2 Differentiability - lecture

GOAL: Define the slope of a function at a point.

Idea: We know what the slope of a line is. Given two points, $x \neq y$ in the domain of function f, we write the slope of the secant line though the two points as $\frac{f(y)-f(x)}{y-x}$. Now we take the limit as $y \to x$ and define this limit to be the slope of the tangent line.

Example: If $f(x) = x^3$ and $x = 1$, then the slope of the secant line between y and x is $\frac{y^3-x^3}{y-x} = \frac{y^3-1}{y-1}$. If we now take the limit as $y \to 1$, we get

$$\lim_{y\to 1} \frac{y^3-1}{y-1} = \lim_{y\to 1}(y^2+y+1) = 3,$$

by simplifying, using the limit laws and direct substitution. Hence, the slope of the function $f(x) = x^3$ at $x = 1$ equals 3.

Definition: A function f is called *differentiable at a point* x if the limit

$$f'(x) := \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists. (Note that f has to be defined and continuous at x.) A function is called differentiable on an open interval, if it is differentiable at each point of the interval. We also write df/dx for the derivative.

Example: When f is a straight line, then this definition of a slope should agree with the slope of the function as we know it. The calculation below shows that this is indeed the case. Let $f(x) = mx + b$ then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0} \frac{mh}{h} = m.$$

Example: For a more complicated example, let's choose $f(x) = \sqrt{x}$ and let's calculate the derivative in general for any $x > 0$ and not for a particular value of x. According to the definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

Now we rationalize the numerator and continue.

$$f'(x) = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}},$$

where the last equality results from direct substitution.

How a function can fail to be differentiable: There are many ways in which a function may fail to be differentiable at a point, corresponding to the ways in which a limit may fail to exist. The following examples stand for various ways in which the failure to be differentiable at a point corresponds to properties of the graph of the function.

1. If the left and right limits exist and are finite but different, then the slope of the function is different, depending on the side form which one approaches the point. We have a corner. An example is $f(x) = |x|$ at $x = 0$. Note that $f(0) = 0$ is defined.

2. If the limit is infinite, then the derivative does not exist. A tangent line must be vertical, and we have a cusp. An example function is $f(x) = \sqrt{|x|}$ at $x = 0$. Note that $f(0) = 0$ is defined.

3. If only one limit (left or right) exists, then the derivative does not exist. For example, the function $f(x) = \sqrt{x^3}$, which is defined only for $x \geq 0$.

4. If the function is discontinuous at x then it cannot be differentiable at x. Example, the function $f(x) =$sign(x) at $x = 0$.

5. If the function has a vertical asymptote, then the function is not differentiable there. Example function $f(x) = 1/x$ at $x = 0$.

Example: Let's do one more abstract example to practice the definition of derivative. Let's find the derivative of a general quadratic polynomial $f(x) = ax^2 + bx + c$.

Derivatives and Graphs: Since we defined derivatives as slopes of tangent lines, there is a nice correspondence between the properties of the derivative and the graph of a function.

$$f'(x) > 0 \Leftrightarrow f \text{ increasing at } x$$
$$f'(x) = 0 \Leftrightarrow f \text{ has horizontal tangent at } x$$
$$f'(x) < 0 \Leftrightarrow f \text{ decreasing at } x$$

Definition: x is called a *critical point of* f if x is in the domain of f and either $f'(x) = 0$ or $f'(x)$ is not defined.

Example: We can find qualitative aspects of a derivative of a function by looking at the graph - and vice versa.

Note: Calculation of derivative from the definition/from first principles can be done in simple cases, see above. But in general, this is a tedious and often tricky way of doing things. Instead, we fill find general rules that we can use to calculate derivatives faster. These rules have two aspects: there are rules of how derivatives behave with respect to function operations (adding, subtracting, multiplying, dividing, composing functions) and there are rules for certain classes of functions (polynomials, exponentials, logarithms, trigonometric functions). We start with some basic rules.

Differentiation rules: If the derivatives exist, then the following rules hold

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ The proof of that uses Binomial coefficients/Pascal's triangle.
2. $(f \pm g)'(x) = f'(x) \pm g'(x)$
3. If $h(x) = f(x) \cdot g(x)$ then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. If $h(x) = \frac{u(x)}{v(x)}$ then $h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

Note: Even if you have seen all these rules in high school, I strongly recommend that you practice at least 5 differentiation exercises from the book every day for a week.

9 Differentiating exponentials, logarithms, and the chain rule

GOAL: Learn how to differentiate these important functions, and compositions of functions in general.

The exponential function:

$$\frac{d}{dx}e^x = e^x.$$

Note: The exponential function $f(x) = ke^x$ for any real number k is the *only* function with the special property $f'(x) = f(x)$.

Examples: Find the derivatives with the rules that we have learned so far.

- $f(x) = e^{3x}$ (use repeated product rule)
- $f(x) = e^{-x}$ (use quotient rule)
- $f(x) = x^n e^{-x}$. This function is related to the Gamma function; it is quite important in statistics.

The Chain Rule: If f, g are differentiable functions, then

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x).$$

Special case: Inverse functions.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Application: Logarithm: The inverse of $f(y) = e^y$ is $f^{-1}(x) = \ln(x)$. Then we calculate

$$(\ln(x))' = \frac{1}{(e^y)'} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Using logarithms in differentiation:
Now that we know the derivative of the natural logarithm, we can use it to differentiate other functions. The important formula is $a = e^{\ln a}$.

- To differentiate $f(x) = \log_a x$, we write $f(x) = \frac{\ln x}{\ln a}$ and differentiate simply $f'(x) = \frac{1}{x \ln a}$.
- To differentiate $f(x) = x^x$, we rewrite $f(x) = e^{x \ln x}$ and use the chain and the product rule.

$$f'(x) = e^{x \ln x}\frac{d}{dx}(x \ln x) = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1).$$

9.1 Practice makes progress

The material is covered in Sections 5.1 and 5.3 of the book (4.3 and 4.4 in the first edition). Suggested exercises are

- second edition: **5.1:** 1–42 (*All* the problems with the exponential function)
- second edition: **5.3:** 1–14, 17–41, 42–57
- second edition: sketching problems **5.3:** 5, 16
- second edition: applications **5.3:** 62–68
- first edition: **4.3:** 1-27, 36-41
- first edition: **4.4:** 1–46

Question 1: For each of the following functions, find the derivative.

$$\text{(a) } f(x) = \ln\left(e^{x^3} \cdot \frac{x^4 - 3x^2 + 17x - 8}{\ln(x)}\right), \qquad \text{(b) } f(x) = \ln\big(1 + x\ln(x)\big)$$

$$\text{(c) } f(x) = \left(\frac{\sqrt{x+1}}{e^{x^2}+5}\right)^8, \qquad \text{(d) } y(x) = \log(10^x(3x^2-1)^3)^5$$

$$\text{(e) } f(x) = \ln(x^{10}), \qquad \text{(f) } h(z) = \frac{5^{-z}}{\sqrt{z}}$$

$$\text{(g) } f(x) = \frac{1}{\sqrt{2}}e^{-(x-10)^2}, \qquad \text{(h) } f(x) = \ln\left(\frac{1}{x^2+1}\right)$$

$$\text{(i) } y(x) = \ln\left(7e^{x^3}(x^3-3)^2\right), \qquad \text{(j) } w(y) = \ln\left(\sqrt{y^2+3y+9}\right)$$

$$\text{(k) } f(t) = \ln\left(\frac{(t^3+1)^7}{(t+2)^6}\right)$$

Question 2: Consider the function $f(x) = e^x x^{-n}$ with parameter $n = 1, 2, 3, \ldots$. Find the equation of the tangent line to the graph of the function at $x = 1$.

10 Sine, Cosine and implicit differentiation

GOAL: Differentiate trigonometric functions and functions that are given implicitly.

Derivatives of some trig functions:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = 1 + \tan^2(x) = \sec^2(x)$$

Note: Mathematical notation is not always consistent. We write $\cos^n(x) = (\cos(x))^n$ for *positive* integers n. But $\cos^{-1}(x)$ stands for the inverse function $\arccos(x)$ and not for the fraction $\frac{1}{\cos(x)}$.

Inverse trigonometric functions: via the chain rule

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(\arccos(x))} = \frac{-1}{\sqrt{1-\cos^2(\arccos(x))}} = \frac{-1}{\sqrt{1-x^2}}.$$

Implicit differentiation: Sometimes we want to find the derivative of a function that is not given explicitly but rather in a complicated equation. We can apply the chain rule and solve for the derivative that we are interested in.

Logarithmic differentiation: When the independent variable appears in the base and in the exponent of a function, then we can take logarithms first (on both sides!) and then differentiate and solve for the derivative that we are interested in.

$$y(x) = f(x)^{g(x)} \quad \Rightarrow \quad \ln(y(x)) = g(x)\ln(f(x)) \quad \Rightarrow \quad \frac{y'}{y} = g'\ln(f) + g\frac{f'}{f}$$

10.1 Practice makes Progress

The material here is covered in Sections 5.4 and 5.5 of the book (4.5 and 4.4 in the first edition). Suggested exercises are

- second edition: **5.4:** 1–31, 32–35, 36–40, 49–56
- second edition: **5.5:** 1–19
- first edition: **4.5:** 1–31, 32–35, 45–48
- first edition: **4.4:** 35–38, 51–57

Question 1: The location (as a function of time) of a car, moving in a straight line, is given by the expression $x(t) = 2t + \sin(2\pi t)$ for $t \in [0, 1]$. What are the highest and lowest values of its acceleration in that time interval?

Question 2: Find the derivatives of the following functions. Do not simplify.

$$f(x) = e^{\arcsin^5(x)+\arcsin(x^5)}, \qquad f(x) = \arctan(\cos\sqrt{x}) + \arccos(\tan\sqrt{x})$$
$$f(x) = \left(\sqrt[3]{\sin^2(x)}\right)^{\arctan(x)}, \qquad y(x) = \sin(e^{\sec(x^2)})$$
$$y(x) = \frac{5x}{(\tan(x^2))^3}, \qquad g(s) = \frac{\cos(5s+8)}{\sin(5s)}$$
$$y(x) = \frac{1}{(\sin(x^2))^2}, \qquad y(x) = \cos(e^{\cos x})$$
$$f(x) = \cos(x)\sin(5x^2+7), \qquad g(x) = \frac{\tan(x)}{e^{7x}x^4}$$
$$h(x) = e^{\cos^3(x)+2\sin^2(x)}, \qquad h(z) = \frac{\sin(z^5)}{\sqrt{z}}$$
$$g(x) = e^{\cos^2(x^3)}, \qquad f(x) = \frac{\tan(1-x^3)}{x+1}$$
$$g(x) = x^{\cos(x)}, \qquad h(x) = \ln\left(x^5\sin(1+\sqrt{x})\right)$$

Question 3: Use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy $e^{4x\sqrt{y}} = y^5$.

Question 4: Use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy $ye^y - xe^x = 0$.

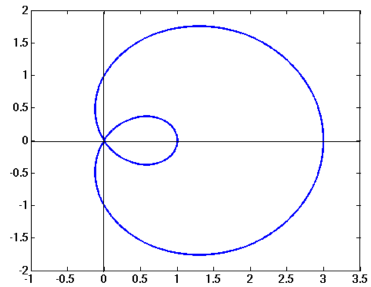
Question 5: Use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy $y\ln(y) - x\ln(x) = 0$.

Question 6: Use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy $2xe^{-x^2} + 3y^2e^y = 5x$.

Question 7: A function $y = f(x)$ is defined implicitly by the equation $x^2y^7 - x^5\ln(y) = 4$.
(a) Find the derivative $\frac{dy}{dx}$
(b) Find the equation of the tangent line to the graph of this equation at the point $(2, 1)$.

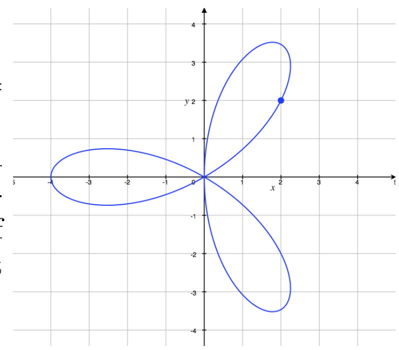
Question 8:

Find the tangent line to the "limax" curve (see plot) at the point (0,1), where the curve is given by the relation $(x^2 + y^2 - 2x)^2 = x^2 + y^2$.



Question 9:

The trifolium is the curve (see plot) given by the equation $12xy^2 - 4x^3 = x^4 + 2x^2y^2 + y^4$.
Identify on the curve of the trifolium the points where we cannot compute dy/dx. Find the equation of the tangent line on the trifolium at the point $(2, 2)$.



11 Second derivative and curve sketching

GOAL: understand curvature, develop a recipe for curve sketching.
Motivation: Consider the two functions $f(x) = e^x$ and $g(x) = \frac{x}{1+x}$ for $x \geq 0$. Both functions have positive first derivatives, hence, both are increasing functions.

$$f'(x) = e^x > 0, \qquad g'(x) = \frac{1}{(1+x)^2} > 0.$$

However, their graphs look quite different. The slope of f increases with x but the slope of g decreases with x. In other words, the function $f'(x)$ is increasing but the function $g'(x)$ is decreasing. Let's calculate this:

$$\frac{d}{dx}[f'(x)] = e^x > 0, \qquad \frac{d}{dx}[g'(x)] = \frac{-2}{(1+x)^3} < 0 \quad x \geq 0.$$

Definition: The second derivative of a function is defined as

$$\frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2}{dx^2}f(x) = f''(x).$$

Second derivatives and graphs: The second derivative measures curvature. If it is positive, then the graph curves upwards (or to the left when thinking about traveling along the graph in positive x-direction). When it is negative, then the graph curves downward (or to the right). A point where the curvature changes from concave up to concave down (or vice versa) is called *point of inflection.* It necessarily has $f''(x) = 0$.

Examples: Power functions $f(x) = x^p$. There are three cases to consider: (i) $p > 1$, (ii) $0 < p < 1$, and (iii) $p < 0$.

Note: The second derivative can also help to identify critical points as (local) minima and maxima. But not always. The same caution is to be exercised with inflection points. Examples are the function $f(x) = x^3$ and $g(x) = x^4$.

Curve Sketching:

1. find the domain of the function
2. find the zeros
3. identify the asymptotes
4. differentiate and find critical points, intervals of increase and decrease
5. differentiate again and find possible inflection points, upward and downward concavity
6. sketch the x-axis and all the information
7. MAKE SURE EVERYTHING IS CONSISTENT

11.1 Practice makes Progress

The material here is covered in Sections 5.6 of the book (4.6 in the first edition). Suggested exercises are

- second edition: **5.6:** 1–8, 11–31, 36–43, 44–47, 66–73
- first edition: **4.6:** 1–8, 11–25, 26–35, 50–57

Question 1: Consider the function $f(x) = \frac{x-1}{x-2}$.

(a) Find the domain of f.
(b) Find the limits of f as x approaches $\pm\infty$
(c) Are there points where f is not continuous? If yes, find the left and right limits there.
(d) Find the intervals where f is increasing and decreasing. Are there critical points?
(e) Find the intervals where f is concave up or concave down.
(f) Draw the graph of f.

Question 2: Consider the function $f(x) = \frac{2x^2+4x+3}{x^2-1}$.
(a) Find the domain of definition of f.
(b) Find the limits of f as x approaches $\pm\infty$.
(c) Are there any infinite limits? If yes, find the left and right hand limit in each case.
(d) Find the intervals where f is increasing and where f is decreasing.
(e) Where does f have a horizontal tangent line?

Question 3: Consider the function $f(x) = \sqrt{3x}\, e^{-x/6}$.

(a) Find the domain of definition.
(b) Find the critical point(s).
(c) Find the intervals where f is increasing or decreasing.
(d) Use a table of values to guess a horizontal asymptote.
(e) Find the intervals where f is concave up or down.
(f) Sketch the graph of the function $y = f(x)$.

Question 4: Consider the function $f(x) = \frac{1}{x^2} + \frac{1}{2x^3}$. Follow these steps to graph the function.
(a) Find the domain of f.
(b) Find the x-intercept(s) of f.
(c) Calculate the derivative of f.
(d) Find the critical point(s) of f.
(e) Calculate the second derivative of f.
(f) Find the point(s) of inflection.
(g) Find the limits $\lim_{x\to 0^\pm} f(x)$.
(h) Find the limits $\lim_{x\to\pm\infty} f(x)$.
(j) Sketch the graph of f for $x \in [-2, 2]$,

Question 5: Graph the function $f(x) = \dfrac{-x^2 + x + 1}{1 - x^2}$ using the following steps.
(a) Give the domain of $f(x)$.
(b) Find the vertical asymptotes.
(c) Find the horizontal asymptotes.
(d) Find the zeros of the function and the y-intercept.
(e) Find the first derivative and simplify it. Does $f(x)$ have any critical points?
(f) Find the inflection points. Indicate where $f(x)$ is concave up or down.
(g) Graph the function $f(x)$, including all the information found above.

Question 6: Use the first and second order derivatives to sketch the graph of $f(x) = \frac{1}{x} - \frac{2}{x^2}$. You have to find the zeros, critical points, inflexion points, intervals where the function is increasing and decreasing, and the vertical and horizontal asymptotes if any.

Question 7: Suppose a function $y = f(x), -\infty < x < \infty$, is continuous, with continuous first and second derivatives. Assume it satisfies the following conditions:

1. $f'(x) < 0$ when $x < 0$, and $f'(x) > 0$ when $x > 0$
2. $f''(x) < 0$ when $x < -2$, and $f''(x) > 0$ when $x > -2$
3. $\lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = 2$.
4. $f(0) = -3, f(-2) = -1$.

(a) Where is the graph of $f(x)$ decreasing?
(b) Where is the graph of $f(x)$ concave up?
(c) Where does $f(x)$ attain a local maximum or minimum?
(d) What are the asymptotes of f?
(e) Sketch the graph of the function $y = f(x)$.

12 Extreme values

GOAL: Identify maxima and minima of functions. (Note that there are different kinds.)

Definition: An *absolute (or global) maximum* of a function f occurs at point c if $f(x) \le f(c)$ for all x in the domain of f. An *absolute (or global) minimum* occurs if $f(x) \ge f(c)$ for all x in the domain of f. We also call global maximum and minimum *global extreme values.*

Examples: Functions may have no global max or min, they may have only one but not the other, they may have both, and such a global max or min need not be unique.

Definition: A *relative (or local) maximum* of a function f occurs at point c if $f(x) \le f(c)$ for all x near the point c. A *relative (or local) minimum* occurs if $f(x) \ge f(c)$ for all x in an open interval around point c. We also call these *local extreme values.*

Examples: Functions may have no local max or min, they may have only one but not the other, they may have both, and a local max or min need not be unique.

Note: A point at the boundary of the domain of the function cannot be a local extremum since we cannot test all values of x near c, but only those in the domain of f.

How do we find extreme values?
Fermat's little theorem says that if f has an extreme value at c and if $f'(c)$ exists, then $f'(c) = 0$. This means that we only have to check critical points. But not every critical point is an extreme value. We need more.

1. If c is a critical point of f and if the sign of the derivative of f changes at c, then a local extremum occurs at c.
2. If c is a critical point of f and if $f''(c)$ exists and is not zero, then a local extremum occurs at c.
3. Note that if $f''(c) = 0$ we cannot conclude whether an extremum occurs at c. Note also that sometimes it is much harder to calculate f'' than to check for a sign change in f'.
4. All points where $f'(c)$ does not exist need to be checked individually.

Recipe for finding extreme values of a function: First find all critical points. Then check each critical point. Finally, compare the function value at critical points and, if applicable, endpoints of the domain of definition.

How do we know that extreme values are there?
The *extreme value theorem* says that a continuous function on a bounded interval has a global maximum and a global minimum. Note that there are two conditions: continuity and bounded interval. If either of them is violated, then the theorem is wrong. This theorem shows once more the importance of the property "continuity".

12.1 Practice makes progress

The material is covered in Section 6.1 of the book (5.1 in the first edition). Suggested exercises are

- second edition: **6.1:** 3–5, 6–21, 22–43, 44, 45
- second edition: **6.1:** 63–68, 81, 82 (Applications)
- first edition: **5.1:** 3–39
- first edition: **5.1:** 47–50 (Applications)

Question 1: Find all local and global maxima and minima of the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{15}{8}x^2 + \frac{2}{3}$ on the interval $[-3, 3]$.

Question 2: Find the global maximum and the global minimum of $f(x) = \sin^2(x)$ on the interval $[\frac{\pi}{4}, \pi]$.

Question 3: For each of the following functions f, find the critical points and classify them; that is find out whether they are local minima, local maxima or neither.
(a) $f(x) = -\dfrac{2}{3}x^3 - 7x^2 - 24x + 14$
(b) $f(x) = 3 - |16 - 8x|$.
(c) $f(x) = x^3 - 6x^2 + 12x - 8$.

Question 4: Find the global maximum and minimum of the function $f(x) = \dfrac{x^2 + 2x}{e^x}$ on the interval $x \in [0, 4]$.

Question 5: Use the first and second order derivatives to sketch the graph of the function $f(x) = \frac{2}{x^2} + \frac{3}{x^3}$. You have to find the zeros, critical points, inflection points, intervals where the function is increasing and decreasing, and the vertical and horizontal asymptotes if any.

13 Optimization

GOAL: Apply our knowledge of extreme values to find "best" values.

Almost all optimization questions include some kind of trade-off. Sometimes, this trade-off is explict and quite obvious, sometimes it is hidden. Uncovering this trade-off is always helpful in finding and interpreting the result.

We practice several word problems here. There is no new theory. Some of these problems are quite simple, some are more abstract, and the last two are quite hard. Not sure that I will get to them in class.

1. Maximization with trade-offs: The yield of crop in agriculture changes with the amount of fertilizer (for example nitrogen) applied. When nitrogen levels in the soil are low, then adding some nitrogen will greatly increase yield. When nitrogen levels are already very high, however, adding more might decrease yield. Assume that yield as a function of nitrogen is given by the equation

$$Y(N) = \frac{N}{1+N^2}.$$

What is the opimal level of nitrogen in the soil?

2. Areas and volumes: Minimize the material used to produce a cylindrical can of a fixed volume.

Approach: Denote by r the radius of the bottom of the can and by h its height. Then the volume is $V = \pi r^2 h$ and the surface area is $A = 2\pi r h + 2\pi r^2$. Use the constant volume condition to replace $h = V/(\pi r^2)$ and minimize the function

$$A(r) = \frac{2V}{r} + 2\pi r^2.$$

Find the critical value of r and the minimal surface area.

3. Distances: Find the distance of the line $y = 1 + 2x$ from the origin and find the point on the line that is closest to the origin.

Answer: The distance of any point (x, y) from the origin is $d = \sqrt{x^2 + y^2}$. If the point is on the line, then $d = \sqrt{x^2 + (1 + 2x)^2} = \sqrt{5x^2 + 4x + 1}$. We need to minimize this function. Equivalently, we may minimize the function

$$f(x) = d^2(x) = 5x^2 + 4x + 1.$$

Since the square root is a monotone function, we will get the same location of extreme values.

4. Optimize food intake by adjusting residence time. Suppose that a bee remains at each flower for a fixed amount of time before it travels to the next flower. If that residence time is small, then the bee might leave valuable nectar behind. If it is large, then it might have depleted all the nectar and lost valuable time to look for the next flower. What is the optimal residence time?

Approach: To answer this question, we need to know how much food the bee collects in t time units. Let's call this function $F(t)$. If the bee takes on average τ time units to fly to the next flower, then the rate of nectar collection is

$$R(t) = \frac{F(t)}{t+\tau}.$$

Typically, F should be a positive, non-decreasing function. An example is $F(t) = t/(t+0.5)$. The bee wants to maximize R. The result is the *marginal value theorem*: the bee should leave the lower if the instantaneous food intake falls below the average food intake.

5. Maximize yield in a DTDS. Assume that a population grows logistically and is being harvested according to the DTDS

$$x_{t+1} = 2.5x_t(1 - x_t) - hx$$

where $h > 0$ denotes the intensity of harvesting. At steady state x^* the yield is $Y(h) = hx^*$. If we harvest very little, then the yield is small. If we harvest lots, then the steady state population is small and therefore the harvest. How should we choose h so that Y is maximized?

6. Optimal age of reproduction. Semelparous organisms (what is this? look it up!) reproduce only once in their lifetime. Typically, they can produce more female offspring as they get older. But if they wait too long, then they might die before they reproduce. What then is the optimal age of reproduction? If we denote by $l(x)$ the probability that an individual lives to age x and by $m(x)$ the average number of female offspring of an individual at age x, then the average annual reproduction is given by

$$r(x) = \frac{\ln(l(x)m(x))}{x}.$$

Use $l(x) = e^{-ax}$ and $m(x) = bx^c$ and find the maximum of r.

7. Optimal clutch size. If an organism produces only few offspring, then each has a high probability of survival; if there are many offspring then the survival probability individually declines (look up: r-versus K strategy!). At how many offspring is the total number of survivors maximized?

Approach: Let R denote the total resources (per adult female) for reproduction and N the clutch size. Then the resource per offspring is $x = R/N$. Denote the survival probability of an offspring with resource x as $f(x)$. This function should be positive and non-decreasing. Then the expected number of surviving offspring is

$$w(x) = Nf(x) = \frac{R}{x}f(x).$$

To maximize $w(x)$, we calculate that $w'(x) = 0$ exactly if $xf'(x) = f(x)$. And if $w'(\hat{x}) = 0$ then $w''(\hat{x}) < 0$ exactly if $f''(\hat{x}) < 0$. For an example, choose $f(x) = \frac{x^2}{x^2+k^2}$.

13.1 Practice makes progress

I strongy recommend that you read through section 6.2 in the second edition of the book. It contains three detailed optimization problems and lots of detailed practice problems for those three. In the first edition, the corresponding material is scattered a little, but there is some at the end of 5.1 For general practice problens, please go to the previous section and look at the practice problems annotated 'Applications'.

Question 1: A company harvests fish at some rate $h \geq 0$. The yield is $Y(h) = h(500 - h)$ tons of fish, the selling price is \$200 per ton. The cost for harvesting at rate h is $C(h) = 1000h(1 + 0.1h)$ in dollars.
(a) Find the expression of the profit P (= revenue - cost) as a function of harvesting rate.
(b) Find the harvesting rate that maximizes profit.
(c) Find the maximum profit.

Question 2: Find the point on the curve $y = \sqrt{x}$ closest to the point (10,0). Hint: minimize the square of the distance from (10,0) to (x, y).

Question 3: In a movie theatre, the screen on the wall is 20 m high and its base is 10 m above eye level. Let θ denote the viewing angle of the screen, that is, the angle $\angle BET$ from the bottom (B) of the screen to the top (T), measured from the vertex of your eye (E). At what distance x from the screen should you position yourself to maximize θ? (from D. Kouba)

Question 4: The size of a population of bacteria introduced to a nutrient can be described by

$$N(t) = 5000 + \frac{30,000t}{100 + t^2}.$$

Find the maximum size of this population for $t \geq 0$.

Question 5: When a patient takes a drug, the concentration of this drug in the blood first increases fairly quickly and then declines again. A function that describes this behaviour is $y(t) = te^{-t/2}$, where $t \geq 0$ is the time in hours after the drug is taken.
(a) How long after drug administration does the drug concentration reach its maximum value?
(b) What is the maximum concentration?

Question 6: Find the point on the parabola $y = x^2$ that is the closest to the point $(1, 2)$ in the cartesian plane.

Question 7: Consider a population that grows according to the logistic updating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS $x_{t+1} = x_t(4 - x_t) - hx_t$.

(a) Determine all equilibria of the DTDS.
(b) Determine conditions on h such that all equilibria are biologically relevant.
(c) Determine conditions on h such that the positive equilibrium is stable.
(d) Determine conditions on h such that the positive equilibrium is unstable.
(e) Determine the value of h that maximizes the yield and state the resulting maximum yield.

Question 8: A golf ball hit with an angle of θ radians and initial velocity of 10m/s will fly for a distance of $d(\theta) = 20.41\sin(\theta)\cos(\theta)$ metres before it lands (neglecting air resistance). Find the angle θ^* between 0 and $\pi/2$ radians that maximizes the distance flown, and find the maximal distance.

Question 9: The oxygen concentration in a lake over a single day is given by the equation

$$C(t) = 10t^3 - 120t^2 + 210t + 12000,$$

where time, $0 \le t \le 24$, is measured in hours. When is the oxygen concentration highest, when is it lowest. What are the maximum and minimum values?

Question 10: When a disease appears in a population, health authorities record the number of infected people. One function that describes this quantity is $y(t) = 80t^2e^{-t}$, where $t \ge 0$ is the time in days and y is the number (in units of thousands of people) of infected people.
(a) At what time will there be the most infected people and how many are there at that time?
(b) When is the number of infected people increasing and when is it increasing?
(c) Identify all points of inflection of the function $y(t)$.
(d) Find any horizontal and vertical asymptotes that may exist.
(e) Draw the graph of the number of infected people as a function of time.

14 L'Hopital's rule

GOAL: Apply derivatives to find indeterminate limits.

Motivation: We know that $\lim_{x\to 0} x^3 = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$. What happens when we multiply, i.e., what is $\lim_{x\to 0^+}[x^3\ln(x)]$? In other words: what is stronger, the cubic going to zero or the logarithm going to negative infinity?

Definition: We say that a limit is of indeterminate form if it can be written as a fraction

$$\lim_{x\to a}\frac{f(x)}{g(x)},$$

and one of the two following cases holds

- $f(x)$ *and* $g(x)$ *both* approach infinity as x approaches a.
- $f(x)$ *and* $g(x)$ *both* approach zero as x approaches a.

Note: The motivational example is about products, but we can always write a product as a fraction, for example

$$\lim_{x\to 0^+}[x^3\ln(x)] = \lim_{x\to 0^+}\frac{x^3}{\frac{1}{\ln(x)}} = -\lim_{x\to 0^+}\frac{-\ln(x)}{x^{-3}}.$$

In the expression in the middle, numerator and denominator approach zero, in the expression at the end, they both approach infinity. L'Hopital's theorem tells us how to calculate indeterminate limits without using calculators.

Theorem: If f, g are differentiable and the limit is indeterminate, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

if the latter limit exists.

Examples:

1. $\lim_{x\to 0^+}[x^3\ln(x)]$
2. $\lim_{x\to\infty} xe^{-x}$ (and more: $\lim_{x\to\infty} x^n e^{-x}$ for $n = 2, 3, ...$)
3. $\lim_{x\to 0}\frac{e^x-1}{x}$ (the derivative of the exponential function at $x = 0$)
4. $\lim_{x\to 0^+}\left(\frac{1}{x^2} - \frac{1}{\tan(x)}\right)$
5. $\lim_{x\to 0} x^x$
6. $\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^x$ (a way to define e^y)

14.1 Practice makes progress

The material is covered in Section 6.4 of the book (5.3 in the first edition). Those sections also talk about "leading behavior", which we don't discuss in this course. Suggested exercises are

- second edition: **6.4:** 17–39
- first edition: **5.3:** 17–39

Question 1: Find the following limits, if they exist, without using sequences of numerical values.

(a) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-2x}\right)$ (b) $\lim_{x\to 0}\frac{\sin(x)-x}{x^3}$

(c) $\lim_{x\to 0}\frac{e^x-1-x}{\cos(x)-1}$ (d) $\lim_{x\to\infty}(x+3)^{1/x}$

(e) $\lim_{x\to 1}\frac{3(x-1)^2}{e^{2x-2}-x^2}$ (f) $\lim_{x\to 0^+}\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]$

(a) $\lim_{x\to 0}\frac{\arctan(x)}{x}$ (h) $\lim_{x\to\pi}\cot^2(x)(x-\pi)^2$

(i) $\lim_{x\to\infty}\frac{e^x-2}{3-2e^x}$ (j) $\lim_{x\to(\pi/2)^+}\frac{\cos(x)}{(x-\pi/2)^2}$

(k) $\lim_{x\to 0}\frac{1-e^x}{1-e^{x/2}}$ (l) $\lim_{x\to\infty}\left(\sqrt{x-1}-\sqrt{x+3}\right)$

Question 2: Find the following limits without using a table of values or a calculator.
(a) $\lim_{x\to\pi/2}\tan(x)\left(x-\frac{\pi}{2}\right)$

(b) $\lim_{x\to\pi/2}\tan(x)\left(x-\frac{\pi}{2}\right)^2$

(c) $\lim_{x\to\pi/2}\tan^2(x)\left(x-\frac{\pi}{2}\right)^2$

(d) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

(e) $\lim_{x\to\infty}(\ln x)^{1/x}$

15 Polynomial approximation

GOAL: Use simple functions (polynomials) to approximate complicated functions.

First answer: Tangent line approximation. Given a function $f(x)$ and a (base) point a, we write the tangent line approximation

$$T(x) = f(a) + f'(a)(x - a).$$

Second answer: Let's generalize this idea and find a function whose value, derivative, second derivative, third derivative,... at a point a agree with the corresponding expressions for f.
Definition: The Taylor polynomial of function f of degree n and base point a is given by

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{6}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

In particular, $T_1(x) = T(x)$ is the tangent line. The notation $f^{(n)}$ denotes the n-th derivative of f. The notation $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$ stands for n factorial.

Secant lines and the mean value theorem
With the tangent line, we take value of a function and its derivative at a single point and try to infer something about the value of the function at a nearby point. Using a secant line, we can instead take the value of a function at two points and infer something about the value of its derivative between these two points. This connection is summarized in the Mean Value Theorem.

Mean Value Theorem: If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) then there exists a value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In other words: there is a point where the slope of the function equals the slope of the line connecting the function values at the endpoints ("average slope").

Rolle's theorem: A special case of the MVT is Rolle's theorem. If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), and if $f(a) = f(b)$ then there exists a value $c \in (a, b)$ with $f'(c) = 0$.

Note: Differentiability is essential for these theorems to hold. Think about the absolute value function $f(x) = |x|$ on the interval $[-11]$. The function is continuous everywhere and even differentiable everywhere except for at $x = 0$. Furthermore, $f(1) = f(-1) = 1$, so that Rolle's theorem (if it applied) predicted that there is a point with $f'(c) = 0$. But there is no such point.

15.1 Practice makes progress

The material is covered in Sections 5.7 and 6.3 of the book (4.7 and 5.2 in the first edition). The book covers a lot more about secant lines and about error estimates. We don't go into these topics much. Suggested exercises are

- second edition: **5.7:** 1–7, 8–13 (tangent line only), 14–19, 24–27, 28–33
- second edition: **6.3:** 7–10, 11–14
- first edition: **4.7:** 1–7, 8–13 (tangent line only), 14–19, 24–27, 28–33
- first edition: **5.2:** 7–10, 11–14

Question 1: Consider the function $f(x) = (3x+5)^{4/3}$.
(a) Find the first, second, and third derivative.
(b) Find the Taylor polynomial T_3 using base point $a = 1$ for the function $f(x)$.
(c) Evaluate the error in the approximation by calculating $|f(0.8) - T_3(0.8)|$ to six decimal places.

Question 2: Consider the function $f(x) = 1/x$.
(a) Use the mean value theorem (MVT) to show that there is a number $c \in [1, 2]$ where the function $f(x) = \frac{1}{x}$ has slope -1/2. Find the value of c.
(b) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[a, b]$ with $0 < a < b < \infty$. What is the value of the derivative that the MVT guarantees exists? At which point $c \in [a, b]$ does it occur?
(c) Now consider the same function $f(x) = \frac{1}{x}$ on the interval $[-1, 1.]$ Since $f(-1) = -1$ and $f(1) = 1$ there should be a point $c \in [-1, 1]$ such that $f'(c) = \frac{1-(-1)}{1-(-1)} = 1$, according to the MVT. Calculate f' and show that no such c can exist? What is wrong in the previous reasoning?

Question 3: Find the Taylor polynomial of degree 3 with base point 4 to approximate the value of $\sqrt{5}$.

Question 4: Find the Taylor polynomial of degree 4 of the function $f(x) = \ln(x^2)$ with base point $a = 1$.

Question 5: Find the Taylor polynomial of degree 3 with base point 0 of the function $f(x) = \tan(x)$. Use this polynomial to approximate $\tan(0.1)$. Compare with the true value.

Question 6: How can you approximate the value $10^{8.1}$? How good is your approximation?

Question 7: (a) Find the Taylor polynomials of degree 1, 2 and 3 for $f(x) = \cos(2x) + x$ with base point $a = \dfrac{\pi}{6}$. (x in radians!) Use these polynomials to approximate $f(0.5)$.

Question 8: (a) Find the Taylor polynomials of degree 1, 2 and 3 for $f(x) = \sin(2x) + x^2$ with base point $a = 0$. (x in radians!) Use these polynomials to approximate $f(-0.1)$.

Question 9: Consider the function $f(x) = 1 + \sin(2x - 2)$.
(a) Use the linear approximation of f to estimate the value of $f(0.9)$.
(b) Justify from the graph of f why the approximation of $f(0.9)$ in (a) is below the actual value.
(c) Use a Taylor polynomial of degree 3 to approximate $f(0.9)$.

Question 10: Consider the function $f(x) = x^{4/3}$.
(a) Find the tangent line approximation at $x_0 = 1$.
(b) Find the Taylor polynomial of degree three at $x_0 = 1$.
(c) Use the tangent line to estimate $1.01^{4/3}$.
(d) Draw a graph of the function and explain why the tangent line approximation underestimates the true value.

Question11: Find the Taylor polynomials of degree 3 and 5 of the function $f(x) = \sin(x)$ around $x_0 = 0$.

Question 12: Consider the function $f(x) = e^{3x}$.
(a) Find the tangent line approximation with base point $a = 0$.
(b) Find the Taylor polynomial of degree three with base point $a = 0$.
(c) Compare the value $f(-0.2)$ with the approximation from the tangent line and from the Taylor polynomial.

Question 13:
(a) Find the linear approximation to $f(x) = e^{2\sin x}$ around $a = \pi$.
(b) Use your answer in part (a) to estimate $e^{2\sin(3)}$.
(c) Find the cubic approximation $T_3(x)$ to $f(x) = e^{2x}$ at $a = 0$.

Question 14:
Use a Taylor polynomial of degree three for $f(x) = x^{1/5}$ to estimate $1.1^{1/5}$ without using a calculator.

16 Stability in nonlinear DTDS and Chaos

GOAL: Analyze nonlinear DTDS.

Recall: A linear DTDS $x_{t+1} = rx_t + c$ with $r \neq 1$ has exactly one fixed point $x^* = c/(1-r)$ and this point is stable when $|r| < 1$.

Motivation: A nonlinear DTDS $x_{t+1} = f(x_t)$ can have several fixed points. We want to find a similar simple criterion to determine the stability of each fixed point without doing cobwebbing.

Idea: Since stability is a local concept (all *nearby* solutions converge to the fixed point) we may replace the function f with its tangent line approximation at the fixed point and check stability for the resulting linear system.

Theorem: Let x^* be a fixed point of the DTDS $x_{t+1} = f(x_t)$, i.e., $x^* = f(x^*)$. Then x^* is stable if $|f'(x^*)| < 1$.

Explanation: Write a solution near the fixed point as $x_t = x^* + y_t$ where $|y_t|$ is small. Then the tangent line approximation gives

$$x_{t+1} = f(x_t) \approx f(x^*) + f'(x^*)(x_t - x^*).$$

At the same time $x^* = f(x^*)$, $x_{t+1} = x^* + y_{t+1}$ and $x_t - x^* = y_t$. So, we get the approximation $y_{t+1} \approx f'(x^*)y_t$. This is a linear DTDS; it fixed point $y^* = 0$ is stable if $|f'(x^*)| < 1$. Now, if $y^* = 0$ is stable, then y_t will converge to zero, which means that x_t will converge to x^*.

Examples:

- The Beverton-Holt model $x_{t+1} = \frac{5x_t}{1+x_t}$ has two fixed points. The larger one is stable, the other unstable.
- The Allee effect model $x_{t+1} = \frac{4x_t^2}{1+x_t^2}$ has three fixed points. Two are stable, one is unstable.
- The model with parameter $x_{t+1} = \frac{ax_t}{1+x_t^2}$ has at most one positive fixed point. What is its stability?
- For which values of r is the positive fixed point of the logistic DTDS $x_{t+1} = rx_t(1-x_t)$ stable?

Beyond Stability: What happens in the logistic model when r is so large that the fixed point is unstable? Use the file `LogisticDTDS.xls` on the course website and try it out. You will be surprised.

16.1 Practice makes progress

The material of this section is covered in sections 6.7 and 6.8 of the second edition of the book (sections of the first edition). Suggested practice problems from the book are

- second edition: **6.7:** 1–12, 13–18, 31, 32, 37, 38
- second edition: **6.8:** 1–4, 9–12, 13–18, 19–26, 29, 30
- first edition: **5.5:** 5–12, 13–20, Applications: 23–26
- first edition: **5.6:** 1–4, 5–8, 9–16

Question 1: Consider a population that grows according to the logistic updating function and is harvested at a linear rate $h \geq 0$. The number of individuals of the species satisfies the DTDS $x_{t+1} = x_t(4 - x_t) - hx_t$.
(a) Determine all equilibria of the DTDS.
(b) Determine conditions on h such that all equilibria are biologically relevant.
(c) Determine conditions on h such that the positive equilibrium is stable.
(d) Determine conditions on h such that the positive equilibrium is unstable.
(e) Determine the value of h that maximizes the yield and state the resulting maximum yield.

Question 2: Consider a population that grows according to the Beverton-Holt updating function and is harvested according to a linear rate h. The number of individuals of the species satisfies the DTDS

$$x_{t+1} = \frac{4x_t}{1 + x_t} - hx_t, \qquad t = 0, 1, 2, \dots.$$

(a) Find the fixed points of this DTDS. [One point should depend on the harvesting rate.]
(b) For which values of h is there a positive fixed point?
(c) Which harvesting rate maximizes the number of individuals harvested at the fixed point?
(d) Is the fixed point with the value of h from part (c) stable? [If you did not get the answer to part (c), use $h = 0.5$.]

Question 3: The density of fish (i.e. number of fish per cubic metre) in a lake is determined by the discrete-time dynamical system $x_{t+1} = \dfrac{4x_t}{1 + 3x_t^2}$, where t is the time in years since the beginning of the observation. Initially, the density is $x_0 = 0.5$.
(a) What will the density be after three years? (4 decimal places are enough)
(b) What is the updating function $f(x)$?
(c) What are the biologically relevant equilibria?
(d) Use the derivative test to determine the stability properties for each of the two equilibria.

Question 4: Consider the DTDS $x_{t+1} = f(x_t)$ for each of the updating functions below.
(a) Find the equilibrium point(s).
(b) Use the derivative test to evaluate the stability of each equilibrium point.
(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3.
(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations.

1. $f(x) = \frac{1+x}{1+x^2}$
2. $f(x) = \frac{5x}{1+4x^2}$
3. $f(x) = rxe^{-x}$
4. $f(x) = \frac{2x}{1+0.1x}$

Question 5: Consider the DTDS $M_{t+1} = M_t e^{r(3-\frac{M_t}{3})}, \quad r > 0$.
(a) Calculate the positive equilibrium.
(b) Determine the values of r for which the positive equilibrium is stable.

Question 6: The number of fish in a lake is determined by the DTDS $x_{t+1} = \dfrac{300x_t}{100 + 0.1x_t}$, where t is the time in years since the beginning of the observation. Initially, there are 500 fish.
(a) How many fish will there be after three years?
(b) Find the inverse of the updating function.
(c) How many fish were there one year before the observation started?
Find the two equilibria x_1^*, x_2^* of this system and determine the stability of each one.

17 Newton's method and the intermediate value theorem

GOAL: Find zeros of functions computationally when an explicit solution is not available.

Motivation: Solve the equation $x = e^{-x}$ for x. There is no explicit solution. What can we do? Consider the function $f(x) = x - e^{-x}$. The solution that we are looking for is a zero of this function. The following questions arise:

1. Does the equation have a solution? Does the function have a zero?
2. How can we find the numerical value, at least as good an approximation as we like?

We will proceed in three steps: (1) The intermediate value theorem is a tool to answer the first question. (2) The bisection method is a method for finding a good approximate solution, based on the theorem. This method is reliable but slow. (3) Newton's method is another method, based on tangent line approximation. It is usually much faster, but may produce errors on occasion.

The Intermediate Value Theorem (IVT): Suppose f is a continuous function on $[a, b]$. Then for every number N between $f(a)$ and $f(b)$ there is a value $c \in [a, b]$ such that $f(c) = N$.

Example above: We have the function $f(x) = x - e^{-x}$. We need to find an interval. After some playing around, we find that if we take the interval $[0, 1]$ then f is certainly continuous on that interval, and furthermore $f(0) = -1 < 0$ whereas $f(1) = 1 - 1/e > 0$. Hence $N = 0$ is a number between $f(0)$ and $f(1)$ and therefore, there must be a value $c \in [0, 1]$ with the property that $f(c) = 0$. In particular, we know that there is a zero of f in the interval $[0, 1]$.

The bisection method: Now that we know that there is a zero in a certain interval (from the IVT), we can use the IVT repeatedly (iteratively) to make the interval in which this zero occurs smaller and smaller. The bisection method works as follows:

1. Want to find a solution of $f(x) = 0$.
2. Find an interval $[a_1, b_1]$ so that the function is continuous on $[a_1, b_1]$ and has a sign change there (and therefore a zero in the interval by the IVT).
3. Evaluate the function at the midpoint of the interval $c = \frac{a_1+b_1}{2}$.
4. If the sign change is in the interval $[a_1, c]$ then we know that the zero is in this interval. We choose $[a_2, b_2] = [a_1, c]$. If the sign change is in the interval $[c, b_1]$ then we know that the zero is in this interval. We choose $[a_2, b_2] = [c, b_1]$.
5. We now have an interval that is half the size of the original interval, and in which there is a zero of our function. Now we repeat steps 3) and 4) as often as we like to make the interval as small as we like. In fact, after n steps, the length of the interval will be 2_n times the length of the original interval.

Example continued: To find the solution of $x = e^{-x}$, we apply the bisection method to the function $f(x) = x - e^{-x}$. We choose $[a_1, b_1] = [0, 1]$. We know that there is a sign change, hence a zero (see above).

1. Now we evaluate f at $c = \frac{0+1}{2}$ and get $f(1/2) = 1/2 - e^{-1/2} = -0.1065 < 0$. Hence, the sign change is in the interval $[1/2, 1] = [a_2, b_2]$.
2. Now we evaluate f at the midpoint $c = \frac{0.5+1}{2} = 0.75$. We find $f(0.75) = 0.2776 > 0$ Hence, the sign change is in the interval $[0.5, 0.75] = [a_3, b_3]$.
3. Now we evaluate f at the midpoint $c = \frac{0.5+0.75}{2} = 0.625$. We find $f(0.625) = 0.0897 > 0$ Hence, the sign change is in the interval $[0.5, 0.625] = [a_4, b_4]$.
4. We repeat as often as necessary to get high precision.

Newton's method: The bisection method works very reliably, but it is slow. One reason why it is slow is that it does not include any other information about the function than its value at certain points. Newton's method includes information about the slope of the function, and often get better results. Newton's method works as follows.

1. Want to find a solution of $f(x) = 0$.
2. Take a first guess at where the zero could be; call it x_1.
3. Calculate the tangent line to the function in x_1 and find the zero of the tangent line. It is given by
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$
4. If things go well, then this zero of the tangent line should be closer to the true zero of f than the first guess. Take x_2 as the new guess and repeat step 3) as often as necessary to get close to the true zero.
5. Note that Newton's method really is a DTDS.

Example continued: To find the solution of $x = e^{-x}$, we apply Newton's method to the function $f(x) = x - e^{-x}$. We choose $x_1 = 0$. Then the first iteration gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{-1}{2} = 0.5.$$

The second iteration leads us to

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5 - \frac{0.5 - e^{-0.5}}{1 - e^{-0.5}} = 0.5663.$$

We continue a bit more

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.5663 - \frac{0.5663 - e^{-0.5663}}{1 - e^{-0.5663}} = 0.5671.$$

Note that after this third iteration, the second digit does not change, and after one more iteration, the first four digits do not change anymore (try it out!). So, Newton's method is much faster than the bisection method, where after 4 steps we have only reduced the original interval length by a factor of 1/16.

Also note that Newton's method uses the derivative in the denominator. So, if we are close to a critical point, then Newton's method could have massive problems. Most actual algorithms used in root finding therefore use some modified Newton's method.

More examples:

- Use both methods to find the value of $\sqrt{2}$ to three digits by setting $f(x) = x^2 - 2$ and looking for a zero. Start with the interval $[1, 2]$ for the bisection method, use the IVT to show that the zero is in this interval. For Newton's method, start with $x_1 = 1$. Compare how long it takes.

- In the previous example (i.e., $f(x) = x^2 - 2$) what happens if you start Newton's method at $x = 0$? What when you start at $x = -1$?

- Use Newton's method to find an intersection of the curves $y = \ln(x)$ and $y = 2x - 4$. Begin with the point $x_1 = 1$ and calculate three iterations.

- Find an approximate solution of the equation $x = \cos(x)$.

17.1 Practice makes progress

The material in this section is covered in chapters 6.3 (IVT) and 6.6 (Newton) in the second edition of the book (sections 5.2 and 5.4, respectively in the first edition). Suggested practice problems are

- second edition: **6.3:** 1–4, 15, 16, 19–22, 23–26, 28–33
- second edition: **6.6:** 1–8, 17, 18, 20, 27, 28, 35
- second edition: **5.2:** 1–6, 21, 22, 25–28, 36, 37
- second edition: **5.4:** 1–8, 17, 18, 25, 26, 33

Question 1: Consider the function $f(x) = e^x - 3x$.
(a) Show that this function has a zero between 0 and 1.
(b) Use Newton's method to find an approximation to this solution. Write down the general formula. Begin with $x_0 = 0$ and calculate x_1, x_2, x_3. Please give 6 decimal places for your calculation.

Question 2: Consider the DTDS $x_{t+1} = f(x_t)$ with updating function as below.
(a) Use the Intermediate Value Theorem to show that there is an equilibrium in the closed interval $[a, b]$.
(b) Use Newton's method to solve for the equilibrium up to four iterations when $x_0 = a$. Please give decimal 4 points for your calculation.

1. $x_{t+1} = \frac{1}{1+x_t^3}$ on $[0, 1]$, when $x_0 = 0$
2. $x_{t+1} = \ln(3 - x_t^2)$ on $[0, 1]$, when $x_0 = 0$
3. $x_{t+1} = \cos x_t - \frac{1}{2}$ on $[0, \frac{\pi}{2}]$, when $x_0 = 0$
4. $x_{t+1} = x_t^4 - 1$ on $[1, 2]$, when $x_0 = 1$
5. $x_{t+1} = x_t^3 + x_t - \frac{1}{x_t} + 1$ on $[\frac{1}{2}, 1]$, when $x_0 = \frac{1}{2}$
6. $x_{t+1} = 3 - e^{x_t}$ on $[0, 1]$, when $x_0 = 0$

Question 3: Consider the functions $f(x) = e^{x/3}$ and $g(x) = 2 - x^2/2$.
(a) Show that the functions intersect in the interval $[0, 2]$.
(b) Use Newton's method to calculate the intersection point. Write down the general formula for Newton's method. Then start with $x_1 = 1$ and do three iterations.

Question 4: The goal of this question is to show that the function $f(x) = x^3 + x^2 + 3x + 2$ for $x \in (-\infty, \infty)$ has exactly one zero. We split this question into a few subquestions.
(a) Use the intermediate value theorem to show that there exists (at least) one zero.
(b) Use Rolle's theorem to show that if there are two (or more) zeros, then there is at least one critical point.
(c) Show that the function f does not have a critical point.
(d) Put all your arguments together to show that the function has exactly one zero.

Question 5: Use Newton's method to estimate the solution of the equation $\sin\left(x+\frac{\pi}{2}\right)-\dfrac{x}{2}=0$ by completing the following steps:
(a) Use the Intermediate Value Theorem to show that there is a solution between 0 and $\frac{\pi}{2}$.
(b) Perform three iterations of Newton's method with the initial value $x_0=\frac{\pi}{4}$ (use 8 decimal places).

Question 6: Consider the equation $x^4=4x^3+1$.
(a) Show that this equation has a solution between -1 and 0.
(b) Use Newton's method to find an approximation to this solution. Begin with $x_0=-1$ and calculate x_1,x_2,x_3. Please give 6 decimal points for your calculation.

Question 7: Apply Newton's method to find the intersection of the curves $y=\ln(x)$ and $y=2x-4$. Do only three iterations of Newton's method, beginning with the point $x_0=1$.

18 Antiderivatives

GOAL: Reverse the process of differentiation: find a function whose derivative is a given function.

Definition: A *differential equation* (DE) is an equation for the derivative of a function. A pure-time DE contains the independent variable of sought function but not the function itself, whereas an autonomous DE contains the function but not the independent variable. (We study only pure-time equations here; autonomous equations will appear in MAT 1332.)

Definition: An *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. We write

$$F(x) = \int f(x)dx$$

and say "F is the integral of f with respect to x." Function $f(x)$ is called the integrand. If $F(x)$ is an antiderivative of $f(x)$ then $F(x) + c$ is also one. We write

$$\int f(x)dx = F(x) + c$$

for the set of all antiderivatives and call it the *indefinite integral*.

Examples: Everything we have already learned about differentiation can be reformulated in terms of integration. For example, we already know the indefinite integral of many functions.

$$\int 1dx = x + c, \qquad \int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + c$$
$$\int \frac{1}{x}dx = \ln(x) + c, \qquad \int e^x dx = e^x + c$$
$$\int \cos(x)dx = \sin(x) + c, \qquad \int \sin(x)dx = -\cos(x) + c.$$

The goal now is to turn all the differentiation rules into integration rules.

Rules for antiderivatives:

Power rule: $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ if $n \neq -1$.

Constant product rule: $\int af(x)dx = a\int f(x)dx$

Sum rule: $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

18.1 Practice makes progress

The material in this section is covered in chapters 7.1 and 7.2 in the second edition of the book (sections 6.1. and 6.2, respectively in the first edition). Suggested practice problems are

- second edition: **7.1:** 1–9, 10–13, 16–23, Applications: 34–38, 40, 41
- second edition: **7.2:** 1–6, 7–24, 25–34, Applications: 41–46
- first edition: **6.1:** 1–9, 10–13, 16–23, Applications: 34–39
- first edition: **6.2:** 1–9, 10–13, 16–23, Applications: 34–39

Question 1: Find the following indefinite integrals.

(a) $\displaystyle\int \frac{(1+\sqrt{x})^2}{x^2} dx$

(b) $\displaystyle\int \frac{(t+1)^2}{2t^3}\, dt$

(c) $\displaystyle\int (x^{-3/4} - x^{3/4}) dx$

(d) $\displaystyle\int (x^3 + x^{1/3}) dx$

(e) $\displaystyle\int \frac{(x+1)^2}{x}\, dx$

(f) $\displaystyle\int \frac{(\sqrt{x}+2)^2}{x^2}\, dx$

(g) $\displaystyle\int \frac{(1-x)^2}{x}\, dx$

(h) $\displaystyle\int \frac{(2-x)^2}{x}\, dx$

(i) $\displaystyle\int \frac{(x+3)^2}{x^2}\, dx$

(j) $\displaystyle\int \left(10x^4 - \frac{2}{x} + \frac{4}{\sqrt[3]{x}} - 1\right)\, dx$

Question 2: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by $f(t) = 3t^3 + 1$.

19 Integration by substitution

GOAL: Learn to integrate (certain) products.

Motivation: Find the integration rule that corresponds to the chain rule for differentiation.

Rule:

$$\int f'(g(x))g'(x)dx = f(g(x)) + c$$

Example: What is $\int 2xe^{x^2}dx$?

Recipe:

1. Define new variable as function of old: $y = g(x)$. (Usually choose the innermost function.)
2. Differentiate the new variable with respect to the old: $\frac{dy}{dx} = g'(x)$.
3. Multiply by dx to get $dy = g'(x)dx$.
4. Write the entire integral in terms of the new variable: $\int f'(g(x))g'(x)dx = \int f'(y)dy$.
5. Integrate: $\int f'(y)dy = f(y) + c$.
6. Substitute back: $f(y) + c = f(g(x)) + c$.
7. Check your calculation!

Examples:

- $\int e^{3x}dx$
- $\int \frac{1}{1+3x}dx$
- $\int \cos(2\pi(x-1))dx$
- $\int \cos(x)e^{sin(x)}dx$
- $\int \frac{e^{-3t}}{(1+e^{-3t})^3}dt$
- $\int \frac{\ln(x)}{x}dx$
- $\int \frac{(4t+2)^2}{t^2}dt$

19.1 Practice makes progress

The material in this section is covered in chapter 7.5 in the second edition of the book (section 6.5 in the first edition). Suggested practice problems are

- second edition: **7.5:** 1–22, 23–35 (evaluate the indefinite integrals only, don't worry about the boundaries of integration), Applications: 80–83, 86, 87
- first edition: **6.5:** 1–22, 23–35 (evaluate the indefinite integrals only, don't worry about the boundaries of integration), Applications: 52–55, 58, 59

Question 1: Find the following indefinite integrals.

(a) $\int \frac{[\ln(x)]^3}{3x} dx$

(b) $\int \frac{3x+1}{(3x^2+2x+1)^6}\, dx$

(c) $\int \frac{t}{(1+t^2)^2} dt$

(d) $\int \frac{e^x}{e^x+1} dx$

(e) $\int \frac{(\ln(z))^2}{z} dz$

(f) $\int \frac{\sin(\frac{1}{x})}{x^2} dx$

Question 2: Find the indefinite integral of each of the following functions. Check your results by differentiating.

1. $f(x) = (17x - 17)^{17}$
2. $f(x) = 5^{2x-3}$
3. $f(x) = \sqrt[3]{12x + 2014}$
4. $f(x) = \frac{1}{x(\ln x)^2}$
5. $f(x) = e^{3x}\sqrt{2 - e^{3x}}$
6. $f(x) = \frac{5}{1+9x^2}$ [Hint: differentiate $g(x) = \arctan(x)$.]
7. $f(x) = \frac{1}{\sqrt{1-16x^2}}$ [Hint: differentiate another inverse trigonometric function.]

Question 3: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by

1. $f(t) = \frac{1}{17t+12}$
2. $f(t) = 12e^{2t}$

Question 5: Find the following indefinite integrals

(a) $\displaystyle\int \frac{\tan(x)}{\ln(\cos(x))}\,dx$

(b) $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$

(c) $\displaystyle\int \frac{\cot(x)}{\ln(\sin(x))}\,dx$

(d) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

(e) $\displaystyle\int \frac{e^x+1}{e^x+x}\,dx$

(f) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$

(g) $\displaystyle\int \frac{\cos(x)}{(\sin^2(x))^{1/3}}\,dx$

(h) $\displaystyle\int \cos(x)e^{\sin(x)}\,dx$

(i) $\displaystyle\int \frac{e^x+2}{e^x+2x}\,dx$

(j) $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$

(k) $\displaystyle\int \left(\frac{2}{x(1+\ln(x))}\right)\,dx$

(l) $\displaystyle\int \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx$

Question 6: Find the anti-derivative $F(x)$ of $f(x) = \dfrac{e^{\arcsin(x)}}{\sqrt{1-x^2}}$ such that $F(0) = 1$.

20 Integration by parts

GOAL: Learn to integrate (certain) products.

Motivation: Find the integration rule that corresponds to the product rule for differentiation.

Rule:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Example: What is $\int xe^x dx$?

Recipe:

1. Identify the integrand as a product of two functions, $u(x) \cdot v'(x)$, so that $u(x)$ can be differentiated and $v'(x)$ can be integrated (easily).
2. Write down $u(x), u'(x), v'(x), v(x)$.
3. Apply the formula.
4. Repeat if necessary.
5. Check your calculations.

Examples:

- $\int \ln(x)dx$
- $\int x^2e^{3x}dx$
- $\int x\ln(x)dx$
- $\int \frac{\ln(x)}{x^2}dx$
- $\int e^x \sin(x)dx$ [tough!]

And to close, an integral that requires both, integration by parts and substitution.

- $\int \sin(\sqrt{x})dx$ [even tougher!]

20.1 Practice makes progress

The material in this section is covered in chapter 7.5 in the second edition of the book (section 6.5 in the first edition). Suggested practice problems are

- second edition: **7.5:** 36–47, Applications: 50, 51, 56, 57
- first edition: **6.5:** 36–47, Applications: 84, 85

Question 1: Find the following indefinite integrals.
(a) $\int (x+1)\sin(x)dx$
(b) $\int x^2\cos(x)dx$
(c) $\int 16x^3\ln(7x)dx$

Question 2: Find the indefinite integral of each of the following functions. Check your results by differentiating.

1. $f(x) = \frac{x}{2}\cos(5x)$
2. $f(x) = \sqrt{x}\ln x$
3. $f(x) = \arcsin(x)$
4. $f(x) = x3^x$
5. $f(x) = x^2e^{-x}$

Question 3: Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by

1. $f(t) = (t+t^2)e^{-t}$
2. $f(t) = 3t\cos(t^2)$

Question 4: Find the following indefinite integrals

(a) $\int x\ln(x)\,dx$

(b) $\int (x+1)\sin(x)\,dx$

(c) $\int (x+1)\cos(x)\,dx$

(d) $\int (x+1)\ln(x)\,dx$

(e) $\int (x-2)\sin(x)\,dx$

(f) $\displaystyle\int \sqrt[3]{x}\ln(x)\,dx$

(g) $\displaystyle\int 3x^2\cos(0.5x)\,dx$

Question 5: Find the function $f(x)$, such that $f''(x) = \ln(x)$ and $f(1) = f'(1) = 0$.

Question 6: Let $V(t)$ be the volume of a benign tumour in cm^3 after t years. For $t \geqslant 0$, suppose that $V(t)$ satisfies the following differential equation

$$\frac{dV}{dt} = (1+t)e^{-t}.$$

a) If initially $V(0) = 1$, find $V(t)$.

b) Compute $\displaystyle\lim_{t\to\infty} V(t)$ and interpret.

c) Use Newton's method to find when the volume of the tumour will be 2 cm^3. Use 5 decimal places in your computations and find the answer with 3 decimal places of precision.

21 Definite integrals and the fundamental theorem of calculus

GOAL: